Approximate Set Theory

Chord Categories, Voicings, and Interval Cycles

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Abstract This article describes an approximate set theory modeling intuitions shared by musicians such as Cowell, Schoenberg, Messiaen, and Persichetti. The author considers five approximation strategies, showing that in each case the result resembles an exact seven-tone set theory. Since most seven-tone sets are interval cycles, approximate twelve-tone sets are approximately cyclic as well. The theory explains how to highlight this cyclic structure using *voicings*, modeled by intervals in the *intrinsic scale* formed from a chord's own notes. This connection to voicing is what gives approximate chord categories much of their significance. The approach is most useful for chords with five or fewer notes and works tolerably for hexachords, but it breaks down with larger collections. This is not a failure of the model but a reflection of the fact that quality space contracts as cardinality increases.

Keywords set theory, voicing, intrinsic scale, analysis

APPROXIMATE INTERVAL CATEGORIES are a staple of informal musical discourse; musicians of all stripes speak of *steps* and *thirds*, or *clustered* and *quartal* harmonies. Approximate terminology has the cognitive advantage of reducing our harmonic taxonomy and the perceptual advantage of reflecting the often imprecise nature of musical experience: rather than requiring (or postulating, or hoping) that listeners maintain an exact tally of all the intervals they hear, generic qualia (e.g., *tertian*, *quartal*) allow for a degree of listener imperfection. Analysts frequently encounter passages saturated with a single type of generic interval, for example, an abundance of major and minor thirds or perfect and augmented fourths.[1] And as we will see, approximate categories highlight compositional affordances that might otherwise go unnoticed.

Yet academic music theory tends to valorize exact relationships. This is most obviously true of musical set theory, which categorizes chords by exact interval content, measured along the diatonic, chromatic, or some other scale. It is also a feature of twelve-tone music, which emphasizes rigid transformations of ordered

1 The distinction between "generic" and "specific" intervals originates with Clough and Myerson 1985, though that work emphasizes scale membership rather than direct categorization of chromatic intervals.

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collections.[2] It is even characteristic of approaches that group sets into larger categories or genera. Ian Quinn (2001) has surveyed a range of such classification schemes, showing that they tend to converge: typically, chromatic intervals are treated as separate and unrelated qualia, or perhaps arranged along an unbroken continuum from small to large. The implicit picture seems to be of a two-stage process: first listeners flawlessly perceive the total interval content in a passage; then they construct categories by judiciously ignoring some of this information.[3]

This article instead begins with generic intervallic categories, such as second, third, and fourth, using these to reconstruct an analogue to traditional set theory. Though the resulting categories are approximate, they still allow for precise analytical observations. For example, I will show that there are just four kinds of approximate trichords: three are cyclic (cluster, tertian, quartal), and the fourth is equally balanced between cycles; each type presents distinct opportunities to a composer or improviser. Meanwhile, there are five kinds of approximate tetrachord: the four trichordal categories along with a smaller class of "noncyclic" outliers. As chords grow, the number of categories shrinks: there are only four kinds of approximate pentachord, and there is little or no difference among clustered, tertian, and quartal hexachords; instead, the clusters become more and more predominant. Larger collections therefore require new strategies, which is why I focus on smaller sets here.

The resulting framework suggests that our conception of chord quality is fundamentally ambiguous. For small chords, terms such as *clustered* and *quartal* can be taken to describe an intrinsic structure that persists regardless of how the set is arranged in register; for larger chords, however, these same terms are best understood as ways of distributing notes. Consider the pitch classes {C, D, E, F, G, A}. These can be arranged as the cluster (C4, D4, E4, F4, G4, A4), the stack of thirds (D4, F4, A4, C5, E5, G5), and the stack of fourths (E3, A3, D4, G4, C5, F5). Musicians can make these different qualia salient depending on what they do with the pitch-class set: hearing the stack of thirds (D4, F4, A4, C5, E5, G5), we may not even realize that we are in the presence of a cluster. This sort of multivalence is unknown in the chromatic universe, where collections are almost never cyclic with respect to multiple intervals; in the approximate universe it is just a fact of life.[4]

It is just here that approximate set theory intersects with a second and seemingly unrelated topic, the theory of voicing, understood as the systematic study of how pitch-class sets can be arranged in register. This is because we can

2 As Milton Babbitt (1960) emphasized, standard twelve-tone operations combine distance-preserving operations in pitch (transposition and inversion) with distance-preserving operations in time (retrograde, which preserves distance between order numbers). This perspective is almost explicit in the visual analogies of Schoenberg (1941) 1975.

3 I have sometimes resorted to similar rhetoric (Tymoczko 2011; Callender, Quinn, and Tymoczko 2008), though usually in a more theoretical context.

4 The sole twelve-tone exceptions are the singleton and the eleven-note set, which is simultaneously a stack of semitones and a stack of perfect fifths. In general, the $(n-1)$-note subset of a completely even n-note chord will be generated by all the n-note chord's generators. For example, Quinn (2006) notes that the ten-tone equal-tempered tetrachord 0246_{10} is simultaneously a stack of two- and four-step intervals (0246_{10} and 2604_{10}). Similarly, the seven-tone equal-tempered hexachord 012345_{7} is simultaneously a stack of seconds, thirds, and fourths.

explicitly describe the voicings that bring out different aspects of a collection's structure. For example, if we want to highlight a pentachord's quartal aspect, then we should voice it in open position, with each note two chordal steps above its lower neighbor; if we want to highlight its tertian aspect, then we should use the (2, 1, 1, 2) voicing (described shortly). The theory of voicing thus provides a powerful tool for understanding the alchemical process by which abstract pitch-class sets are transmuted into concrete musical objects. Indeed, voicings can be viewed as constitutive of our approximate categories: what makes a set "quartal" is that it is nearly evenly spaced when in the appropriate voicing.

For small chords it is possible to identify clustered, tertian, or quartal structure at a glance. As we add notes, the proliferation of possibilities makes things harder: given a random pentachord or hexachord, it can take effort to figure out whether it can be shaped into a chain of thirds or fourths. This article therefore develops heuristics for determining a chord's generic affiliations. For example, quartal chords of every cardinality distribute their notes into two approximately equal clusters approximately a tritone apart. Likewise, tertian pentachords tend to have four notes spaced as a cluster, with the fifth about a third away from its neighbors. These heuristics obviate the need for extensive memorization or complex music-theoretical computations.

Approximate set theory begins in the commitment to a loosened conception of musical identity. This loosening helps us understand the music of composers who navigated the chromatic universe using tonal tools—for instance, diatonic genera such as step, third, and fourth. Figures such as George Perle, Pierre Boulez, Milton Babbitt, and Allen Forte rebelled against this approach, developing the more rigorous and purely chromatic discipline that came to be known as posttonal theory. Approximate set theory argues that something was lost in the process: we can translate the earlier discourse into purely chromatic language, reconceiving seconds, thirds, and fourths as small, medium, and large chromatic intervals. The resulting categories are doubly advantageous. Analytically, they reflect the practice of a wide range of musicians; compositionally, they provide tools for grappling with the otherwise overwhelming abundance of chromatic possibilities.

Approximate set theory also offers new techniques for recapturing traditional set theory's lost precision. The most important of these involves a new form of exact set theory that measures intervals along the intrinsic scale formed by the notes of a set itself. This gives a precise theoretical language for describing voicing, or the distribution of pitches in register, which in turn allows analysts to understand how composers highlight various intervallic features of their material. These pitch relationships, I argue, are often as interesting as the disembodied pitch-class relationships central to posttonal theory. This focus on pitch structure goes hand in hand with a loosened conception of musical identity, being insensitive to small perturbations in set-class structure.

My thinking here has been influenced by two friends and colleagues. The first is Rudresh Mahanthappa, a deep musical thinker and one of the world's great improvisers. For the past several years he and I have co-taught a course, titled

"Composition and Improvisation," in which we consider sets from tonal, atonal, compositional, and improvisational perspectives. Some of the ideas in this article come directly from Mahanthappa (e.g., Figure 8.2), while others are more indirectly influenced by our collaboration. Above all, he showed me that set theory is not just a pen-and-paper discipline associated with a particular atonal aesthetic, but a living tradition encompassing a wide range of music both composed and improvised, tonal and nontonal.

Equally important is the work of Ian Quinn. Almost all the topics in this article were the subject of intense discussion when we were collaborating with Clifton Callender on voice-leading geometry. I confess that I was initially puzzled by Quinn's ideas, understanding them in a Platonic spirit I could not share. Over the past few years, however, I have been surprised to find myself retracing paths Quinn had already walked along. The result is in many ways an alternative realization of the vision laid out in his "General Equal-Tempered Harmony"—a kind of music-theoretical remix, recombining Quinnian ideas about interval cycle, fuzzy resemblance, and set-class categorization (Quinn 2001, 2006, 2007). Like Quinn, I want to augment the isolated points of classical set theory with regions in a continuous quality space; like Quinn, I consider interval cycles to be prototypical collections supporting flexible categories of the sort that antedated set theory. Quinn's work is more abstract and a priori than mine, aspiring to provide a broad framework applicable to every chord-and-scale environment; by contrast, I am more interested in specific compositional affordances native to the twelve-tone universe. The result might be considered a pedagogical, keyboard-harmony response to ideas that Quinn treats more speculatively—the product of my experience teaching set theory to improvisers.

1. Voicing

A *concrete voicing* is an ordering of pitch classes in register, such as (C3, G3, E4). If we limit our attention to voicings with less than an octave between adjacent notes and decide not to care about the octave in which the voicing appears, then we can identify concrete voicings by listing their pitch classes in ascending order. A list like C–G–E then determines the open-position voicing (C3, G3, E4) up to octave transposition. This bit of shorthand suggests that voicings are closely analogous to twelve-tone rows: both are orderings of an aggregate, with voicings ordering a chord in register and twelve-tone rows ordering the chromatic scale in time. (The appendix explains this connection in detail.) Composers have sometimes expressed twelve-tone rows as voicings, but the more common strategy is to use voicing to add intuitive structure distinct from, but formally analogous to, that of the underlying row.[5]

5 For registrally ordered twelve-tone rows, see Stucky 1981; Perle 1985: 9–10; Mead 1994: 154, 249; Klein 1999; and Capuzzo 2007. The analogy between twelve-tone rows and voicings is implicit in the work of Kholopov, discussed by Segall 2018.

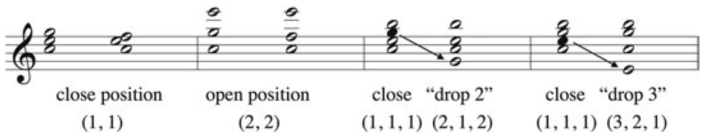

Figure 1.1. Close-position, open-position, "drop-2," and "drop-3" voicings expressed as patterns of steps along the intrinsic scale.

I use the term *concrete voicing* to describe specific arrangements of pitches, whereas I use the more general term *voicing* to refer to patterns of intervals between adjacent notes in a chord. Here we have a variety of options depending whether we measure exactly or approximately and depending on whether we measure along the chromatic scale, the diatonic scale, or some other collection. This article uses three different systems. The first uses approximate intervals measured along the chromatic scale; in this system, "quartal" voicings are those in which each note is about five semitones above its lower neighbor. The second uses exact intervals measured along scales such as the diatonic or harmonic minor; here "quartal" voicings are those in which each note is three scale steps above its lower neighbor. The third system uses exact intervals measured along what I call the *intrinsic scale*, the octave-repeating scale formed from a chord's own notes.[6] Though unfamiliar, this last approach formalizes an important aspect of informal musical discourse: a close-position voicing is one in which every note is one intrinsic step above its lower neighbor; an open-position voicing is one in which every note is two intrinsic steps above its lower neighbor; and the guitarist's drop-2 voicing is the pattern of intrinsic steps (2, 1, 2) produced by displacing (or "dropping") a close-position tetrachord's second-highest note down by an octave (Figure 1.1).[7] Since each pattern can start on any note, an *n*-note chord has *n* different registral inversions of each of its voicings, each with a different chordal element in the bass.[8]

This article explores the convergence of these approaches. For example, a quartal pentachordal voicing, which is to say, a voicing in which each note is approximately five chromatic semitones above its lower neighbor, will be in open position, which is to say that each note will be exactly two intrinsic steps above its lower neighbor. Furthermore, quartal pentachords can typically be

6 The intrinsic scale is a central topic in my recent work (Tymoczko 2020b and 2023).

7 That is, the drop-2 voicing G3 C4 E4 B4 can be conceived as **G** *b* **C E** *g* **B**, skipping steps between G and C and between E and B; this is the pattern (2, 1, 2). These voicings, by virtue of being nearly quartal, are easy to play on the guitar (Tymoczko 2023). Though common in the pedagogical literature, this approach to inversion is not widespread among academic theorists; Vincent Persichetti (1961: 101, ex. 4–24), for example, constructs inversions by placing a chord's bass note in the soprano.

8 My ideas here intersect with two very different music-theoretical traditions. One is practical and focused on improvisation (e.g., Levine 1989; Laukens 1995; Bicket 2001; Herrlein 2011); it tends to describe intrinsic spacing using generative language (e.g., "take a close-position tetrachord and 'drop' the second-highest note by octave"). The other is smaller and more academic, exploring the relation between pitch and pitch-class sets (e.g., Bernard 1987; Morris 1995). Intermediate between these traditions are Cowell (1930) 1996; Persichetti 1961; Ulehla 1966; and Harrison 2014, 2016.

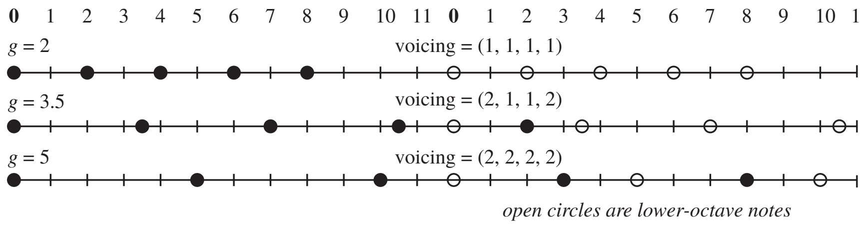

Figure 1.2. Generated pentachords when the generating interval *g* is 2, 3.5, and 5. The first produces a close-position voicing, the second produces the voicing (2, 1, 1, 2), and the third produces the open-position voicing.

embedded in familiar scales as 01245_7 sets, measuring in scale steps, as with the diatonic set CDE•GA. (I use subscripts to identify the size of the scale containing nonchromatic sets; 01245_7 means "the 01245 set in some contextually determined seven-note scale.") Understanding this convergence is useful, first because it helps us recognize a chord's quartal potential when it appears as in close position (i.e., as a 01245_7 set in some familiar seven-note scale), and second because it tells us how to arrange that chord to bring out its quartal qualities—that is, as the open position voicing $\hat{2}$–$\hat{5}$–$\hat{1}$–$\hat{4}$–$\hat{0}$, labeled in diatonic scale degrees.

Here it is helpful to imagine a "generated pentachord": a stack of five notes, each *g* semitones above its lower neighbor, with *g* being any interval whatsoever, including fractional values such as 3.5 (three and one-half semitones, seven quarter tones, or 350 cents). When the generating interval *g* is small, say, one or two semitones, then the chord is a cluster spanning less than an octave; its cyclic nature can be expressed by a close-position voicing in which each note is one intrinsic step above its lower neighbor. But now suppose that *g* is between three and four semitones; in this case the top note will be more than an octave above the first note but less than an octave above the second (Figure 1.2), and the chord's cyclic nature will be expressed by the intrinsic voicing (2, 1, 1, 2). Unlike the cluster, the cyclic structure may not be obvious in close position, where it will have four notes close together, with an outlier at some distance from the others. Finally, when *g* is large, say, five semitones, then two notes will lie in the upper octave and the cyclic structure will be most obvious in open position, with each note two intrinsic steps above its lower neighbor. These voicings are all evenly spaced, with exactly the same distance separating each note from its registral neighbors (Morris 1987: 54).

Already we have encountered a situation where exact chromatic thinking can be detrimental, for in twelve-tone equal temperament there are no tertian five-note interval cycles; this is because repeated major or minor thirds produce note duplications, as in D3–F3–A♭3–B3–(D4) or C3–E3–G♯3–(C4)–(E4). As a result, the systematic relation between voicing and interval cycle can escape our notice. Things become clearer when we consider either fractional distances like

<table>
<tr><th></th><th>3</th><th>4</th><th>5</th><th>6</th><th>7</th></tr>
<tr><td>$g < 2$</td><td rowspan="5">11 **CTQ**</td><td rowspan="4">111 **CT**</td><td rowspan="3">1111 **C**</td><td rowspan="2">1111 **C**</td><td>1111 **C**</td></tr>
<tr><td>$2 < g < 2.4$</td><td>21112</td></tr>
<tr><td>$2.4 < g < 3$</td><td>21112</td><td>22122</td></tr>
<tr><td>$3 < g < 4$</td><td>2112 **T**</td><td>22122 **T**</td><td>222222 **T**</td></tr>
<tr><td>$4 < g < 6$</td><td>212 **Q**</td><td>2222 **Q**</td><td>23232 **Q**</td><td>333333 **Q**</td></tr>
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Figure 1.3. Cyclic voicings for chords of size 3–7, with generating intervals. C, T, and Q stand for the clustered, tertian, and quartal voicings.

3.5, or approximate intervals like "third" or "fourth."[9] A chord like C–E–G–B–D can thus be conceived as an equal-tempered approximation to a musical possibility native to other scale systems; approximate interval categories give us access to this possibility without requiring that we leave twelve-tone equal temperament.[10]

For every chord size there are cyclic voicings that highlight particular intervallic qualities. These are the patterns of intrinsic steps produced by exact interval cycles. Figure 1.3 lists the cyclic voicings for chords with three through seven notes, labeling the clustered, tertian, and quartal voicings. (Beyond that, we have quintal, sextal, and septimal voicings, which, as the appendix explains, can be derived from the quartal, tertian, and clustered voicings.) Even if a chord is not exactly quartal, the quartal voicing will make it as fourthy as possible; for some chords, this will be exactly quartal, for some it will be nearly quartal, and for others it will not be particularly quartal—but then there will be no better alternative.[11] For example, Figure 1.3 tells us that the tetrachordal tertian voicing is (1, 1, 1) while the quartal voicing is (2, 1, 2). Putting the French sixth into the former position gives B–D♯–F–A, which is a stack of thirds with a single diminished third (a "near third"); putting it into the latter position gives F–B–E♭–A, which is a stack of augmented fourths with a four-semitone diminished fourth (a "near fourth"). Though the French sixth is not exactly tertian or quartal, these voicings reveal that it is reasonably close to being both.

This sort of knowledge is straightforwardly useful to composers and improvisers: given a collection, it shows us how to highlight various aspects of its intervallic structure. It also gives analysts a tool for examining how composers make use of the opportunities available to them. Figure 1.4 shows that both Bill Evans's "So What" chord and Arnold Schoenberg's "Farben" chord are open-position pentachords, dividing two octaves into five parts: Evans's chord is almost a stack of perfect fourths, with just one major third; Schoenberg's chord is not obviously quartal, containing

9 On average, such chords will have a distance between adjacent notes that is between 3 and 4.

10 Yust 2015 considers equal-tempered music as the manifestation of continuous structures existing outside of equal temperament; this perspective is also common in work on tuning and intonation (e.g., Sethares 1999).

11 See the appendix for more discussion, as well as a catalog of voicings.

	CHR	DIA	INTR
F–A	4	2	2
C–F	5	3	2
G–C	5	3	2
D–G	5	3	2

	CHR	MM	INTR
E–A	5	3	2
B–E	5	3	2
G♯–B	3	2	2
C–G♯	8	4	2

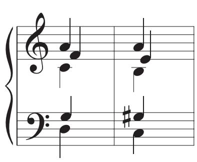

Figure 1.4. The "So What" and "Farben" chords as open-position pentachords. Above, the voicing's intervals are measured in chromatic semitones (CHR), scale steps (diatonic and melodic minor; DIA), and intrinsic steps (INTR).

a minor sixth (a triply augmented fourth) and a minor third (a doubly diminished fourth). But it does contain the quartal segment B–E–A, and on average its intervals are close to quartal; given the notes G♯–A–B–C–E, this is about the best one can do. Were the G♯ lowered by two semitones, the chord would be recognizably quartal.

We can turn one open pentachordal voicing into another by moving the top note down by two octaves; this is because, for any pentachord, taking five two-step motions along the intrinsic scale is equivalent to moving by two octaves. Moving the chord's top note down by two octaves is thus equivalent to moving the entire configuration down by two intrinsic steps. In the case of the "So What" chord, this yields a stack of four perfect fourths, A2–D3–G3–C4–F4. In the case of the "Farben" chord it produces A2–C3–G♯3–B3–E4, which is again not particularly fourthy. But moving the E down by an octave gives us a stack of thirds A2–C3–E3–G♯3–B3, with intrinsic spacing (2, 1, 1, 2). Here we learn that the "Farben" chord is a late-Romantic ninth chord revoiced in a characteristically quartal way. This revoicing is not entirely smooth, and the result is not exactly quartal—the voicing is like a suit that, while not fitting perfectly, is passable at a distance. In much the same way, one could voice the "So What" chord using the tertian pattern (2, 1, 1, 2), yielding F3–A3–C4–D4–G4.[12] This is the "Farben" chord's converse, a quartal collection wearing thirdy clothes.

There is an important difference between the "So What" chord and the stack of thirds A2–C3–E3–G♯3–B3. The "So What" chord is a stack of two-step intrinsic intervals, each of which is very close to a perfect fourth: chromatically, it divides two octaves nearly equally; intrinsically, it divides two octaves exactly evenly. This means that its open-position voicings are all quartal or nearly so. By

12 Readers can use the website https://www.madmusicalscience.com/voicing.html to calculate and explore chord voicings.

Figure 1.5. Eddie Harris's "Freedom Jazz Dance" opens with an octave-displaced open-position pentatonic voicing; the rest of the phrase hints at two other open-position pentatonic voicings.

contrast, the chord A2–C3–E3–G♯3–B3 combines one- and two-step intrinsic intervals and does not divide any number of octaves nearly evenly. This means that its various (2, 1, 1, 2) voicings are not equally tertian; for example, if we start the (2, 1, 1, 2) pattern on E rather than A, we get E3–A3–B3–C4–G♯4, which is not at all tertian. In this respect the "Farben" chord is typical: as a rule, there will be one specific registral inversion in which a chord's cyclic quality is clearest. The "So What" chord is special insofar as it is both a maximally even chord (dividing the octave into five parts, as evenly as possible in the twelve-tone universe) and voiced in a completely even way, with each note the same number of intrinsic steps above its lower neighbor. This produces the unusual situation in which the open-position voicings are all nearly quartal.

There is also a general difference between trichords and pentachords, on the one hand, and tetrachords and hexachords on the other. Since 3 and 5 are prime numbers, every nonzero intrinsic interval generates a complete cycle, touching on every note in the collection. For even-cardinality chords there is no open-position cyclic voicing, as a series of two-step intervals sounds only half the chord's notes.[13] Instead the cyclic voicings include adjusted intrinsic interval cycles that avoid repetition by perturbing the generating intrinsic interval. For example, the quartal voicing E–A–D–G can be viewed as an almost open-position voicing whose middle interval is one step too small: quartal tetrachords like **E** *g* **A D** *e* **G** use the voicing (2, 1, 2) rather than (2, 2, 2). In much the same way, the tertian hexachordal voicing is an almost-open voicing whose middle interval is one step too small: eleventh chords like **D** *e* **F** *g* **A C** *d* **E** *f* **G** use (2, 2, 1, 2, 2) rather than (2, 2, 2, 2, 2).

Though *voicing* has harmonic connotations, the concept can be useful in the melodic domain as well. Figure 1.5 shows that Eddie Harris's "Freedom Jazz Dance" begins by arpeggiating the open voicing of the pentatonic scale, very much like Evans's "So What" chord. Since a unidirectional arpeggiation would be musically awkward, spanning two octaves and not sounding melodic, Harris adds a pair of octave displacements to bring the entire figure within an octave; this produces a close-position voicing in register even while the melodic intervals arpeggiate two-

13 Again, I use *open position* to refer to a voicing whose intervals are all two intrinsic steps: (2, 2, . . . , 2).

Figure 1.6. The first theme of Mozart's Symphony no. 40, I, K. 550, opens with an octave-displaced tertian voicing of the G melodic minor scale.

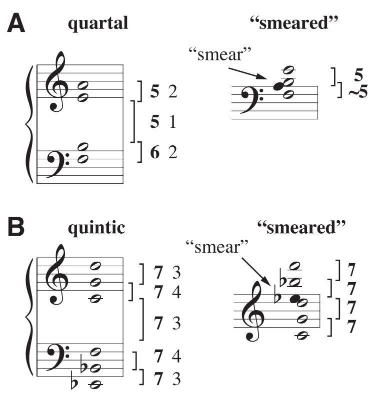

Figure 1.7. Using octave displacements to create "smeared" quartal and quintic voicings. A. A rootless G^7 voicing. B. A quintal hexachordal voicing.

step intrinsic intervals. A similar point could be made about the many functionally tonal melodies that outline octave-displaced tertian voicings of the diatonic scale (Figure 1.6). In these passages voicing becomes a melodic phenomenon.

Octave displacements sometimes appear harmonically as well. Figure 1.7A shows a rootless G^7 voicing that displaces the top note of a fourth-stack down by an octave; the result is an approximate fourth stack in which one chordal slot is occupied by a stepwise pair, a "blurred" or "smeared" note. I find the quartal quality audible despite the smearing. Figure 1.7B applies a similar distortion to a quintal hexachordal voicing, spaced (3, 4, 3, 4, 3) and spanning almost three octaves. Moving the bottom three voices up by two octaves produces the "smeared" quintal voicing (4, 3, 1, 3, 4); it is common in jazz and can be heard in Steve Reich's *Nagoya Marimbas* (mm. 57–64; see also Levine 1989: 141). Once again I find the quintic quality perceptible despite the smearing. As a general rule we can take any stack of large intervals (i.e., fourths or larger) and transpose part of it to produce a "smeared" interval stack.

Another important type of voicing is the "gapped stack," a collection of pitches that could be made cyclic with the addition of a single note. In many cases these voicings have the quality of the larger stack; for example, the chord A3–C4–G4–B4 sounds tertian even though it is intrinsically a cluster (GABC). Here we have an interesting conflict between two different aspects of musical structure: the abstract intervals relating the pitch classes ("cluster") and its

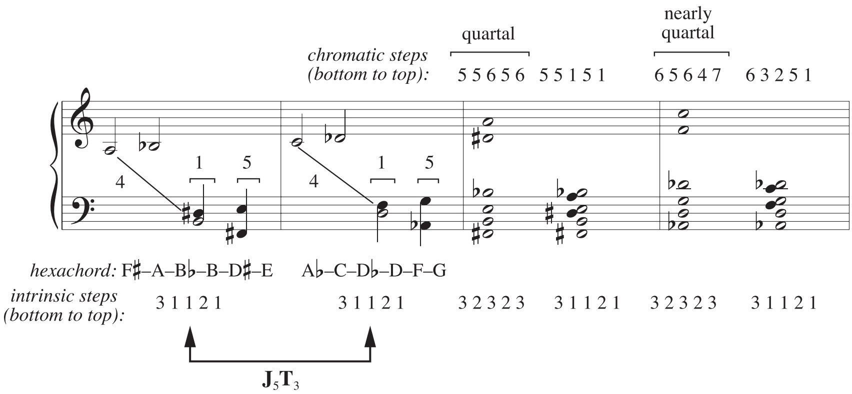

Figure 1.8. Schoenberg's Violin Concerto op. 36 opens with a pair of inversionally related hexachords voiced in the same way. This generalized neo-Riemannian voice leading preserves the relative arrangement of the chromatic cluster (open note heads, left two measures). Each hexachord is voiced as an octave-displaced quartal or nearly quartal voicing. J_5 is the neo-Riemannian voice leading preserving the chromatic cluster (shown with open note heads).

arrangement in pitch ("ninth chord"). Atonal set theory tends to focus on the former, whereas tonal theory often emphasizes the latter.[14] One of the goals of this article is to bridge these perspectives—considering voicing in atonal music, and abstract pitch-content in tonal contexts.

Finally, as I have argued before, the intrinsic scale provides a natural generalization of neo-Riemannian transformations (Tymoczko 2020b). A generalized neo-Riemannian voice leading is nothing more and nothing less than a progression between two inversionally related chords voiced similarly—that is, chords related by pitch-class inversion but spaced in the same pattern of intrinsic steps. Simple mathematics shows that these progressions invariably preserve the distance between at least two voices. The resulting voice leadings link chords that are doubly similar, sharing the same abstract pitch-class intervals (because they are related by pitch-class inversion) and the same intrinsic pitch intervals (because they are voiced similarly). Such voice leadings will arise whenever composers try to find similar-sounding voicings of inversionally related collections.

Figure 1.8 uses these ideas to analyze the opening of Schoenberg's violin concerto. Each hexachord uses the same voicing, a distorted fourth stack in which the top two notes are lowered by an octave. The music resembles Figure 1.7, but instead of a simultaneous smearing, the stepwise pairs become small melodic cells. I find the somewhat fourthy quality of the passage to be both audible and characteristically Schoenbergian.[15] Since the two hexachords use the same

14 See Persichetti 1961: 78–79 for the suggestion that one omit the ninth chord's fifth. The implication is that tertian structure is not destroyed by this omission.

15 Cowell (1930) 1996: 113 describes Schoenberg's fondness for quartal chords.

voicing, they are related by a generalized neo-Riemannian transformation, in this case preserving the spacing of the chromatic cluster. I doubt that Schoenberg was counting hexachordal steps or thinking about the intrinsic scale; instead, I imagine he tried to space his two hexachords similarly, in a way that satisfied his quartal aesthetic. What is interesting is that this sort of intuitive reasoning leads to the precise relationships we have been exploring.

2. Categories

This section models the approximate nature of music perception by grouping chromatic set-classes into larger categories. This is among the most challenging projects a theorist can take on, as it is both intuitively meaningful and yet vague enough to resist easy formalization. Our job is to devise an approach that honors intuition while also adding some useful degree of specificity—yet not too much specificity, as we might want to make room for the thought that some collections could be "sort of" tertian, "somewhere between tertian and quartal," and so forth. Approximate categories are flexible heuristics, useful in some situations but not others.

I will focus on three main categorization schemes: one based on grouping chromatic intervals, one based on scale membership, and one based on quantizing to the nearest seven-tone equal-tempered ("equiheptatonic") set-class. Since I need to refer to these methods repeatedly, I will nickname them the *chunking*, *scalar*, and *quantization* approaches. I will also show that voicing provides a fourth route to approximate set theory, clarifying the musical significance of the others; the appendix discusses a fifth strategy based on the Fourier transform.

My main result is that in lower cardinalities these strategies converge, in each case making approximate twelve-tone set theory look very much like exact seven-tone set theory. Though this might initially seem counterintuitive, it makes sense on reflection: as I discuss below, the seven-note universe is about half the size of the twelve-note universe, and coarse-grained views of chromatic space are going to look similar insofar as they satisfy a few intuitive constraints (e.g., that they have roughly half as many intervals as the chromatic scale, or that chord-categories represent connected regions in voice-leading space). Here we encounter a theme central to Ian Quinn's work: the approximate equivalence of different methods of categorization.

First method: Chunking

The simplest strategy is to group chromatic intervals into classes: instead of conceiving A–C–E–G♯ as a stack of major and minor thirds, we take it to be composed of intervals belonging to a larger class, the third. Here the approximate interval is a kind of musical genus containing two separate species: three semitones and four semitones.[16] There are eleven chromatic intervals between unison and octave. If

16 This approach has been explored in an insightful article by Andrew Mead (1997–98).

we want to group them into roughly half as many categories, we have a choice between five categories, with one larger than the rest, or six categories, with some overlap. The question is whether to adopt the intervals of the six-note, five-interval whole-tone scale or the seven-note, six-interval equiheptatonic scale. This in turn determines whether fourths and fifths are equivalent, like minor and major thirds, or octave complements, like thirds and sixths.

The latter strategy is more familiar: one- and two-semitone intervals are "seconds," with sevenths their complements; three- and four-semitone intervals are "thirds," with sixths their complements; and five- and six-semitone intervals are "fourths," with fifths their complements. Fourths and fifths share the tritone, which can play different roles depending on context. For instance, we classify C–F–B–E as quartal, composed of five- and six-semitone intervals, while we classify E–B–F–C as quintic, composed of six- and seven-semitone intervals. Fourths and fifths together span two semitones (5–6, 6–7), whereas other adjacent categories, such as second and third, span three (1–2, 3–4). The six-note, five-interval strategy instead groups fourths and fifths into a single category, considering five-, six-, and seven-semitone intervals to be "near tritones." From this standpoint, the chord C3–G3–C♯4–F♯4 is a stack of near tritones, with each interval bisecting the octave into two nearly equal halves.

In this article I use the seven-note, six-interval strategy because it is more familiar, more precise, and more analytically fruitful.[17] Furthermore, its interval categories are similar not just in their absolute size (as measured in semitones or cents) but also in their acoustic quality (Cowell [1930] 1996). Figure 2.1 summarizes the relationship. Thirds are sixths are imperfect consonances, having a distinctive phenomenological quality I experience as softness: discounting factors of two, their just-intonation equivalents use the ratios 5/1 and 5/3. Seconds and sevenths are dissonant by virtue of their proximity to the unison and octave.[18] The fourth and fifth are anchored by perfect consonances with the very simple ratio 3/1 and having the phenomenological character of emptiness or hardness; they are joined by the categorially anomalous tritone, which has the most complex frequency ratio of any interval in the just-intonation chromatic scale. This (diabolical) exception notwithstanding, there is substantial overlap between a classification based on approximate semitonal size and one based on consonance. These acoustic facts add an extra dimension of meaning to categories that can be justified on entirely separate grounds.[19]

17 Of course, the seven-note, six-interval system is familiar because it recalls the intervals of the diatonic scale, though here it arises for purely chromatic reasons. In principle, this system is no more diatonic than the six-note, five-interval system is whole tone.

18 They also involve more complicated ratios, but this is possibly irrelevant: we can find arbitrarily complicated ratios that are infinitesimally close to the perfect fifth and hence sound consonant.

19 It is possible to extend the approximate approach to other interval systems. One interesting choice is the pentatonic scale's four-interval system, which divides the chromatic octave into steps of one, two, or three semitones, leaps of four, five, and six semitones, and their complements. This more coarse-grained system cuts across the acoustic distinctions in Figure 2.1, placing the imperfect consonances in different categories.

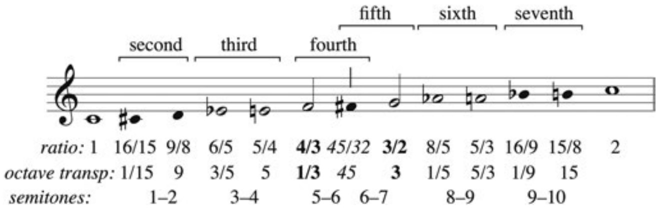

Figure 2.1. Approximate interval categories and consonance.

When categorizing chords in this way, we have a technical choice between *stack equivalence* and *complete generic equivalence*. Two *n*-note chords are stack equivalent if they can be arranged to form the same sequence of *n* – 1 generic intervals. From this point of view, B–D–F–A♭, C–E–G–B, and C–E–G♯–B are all tertian, as each can be expressed as a stack of three thirds.[20] However, not all their intervals are generically equivalent: B–A♭ is a sixth while C–B is a seventh, and C–G is a fifth while C–G♯ is a sixth. The more stringent criterion of complete generic equivalence requires approximate equality of all intervals.[21] I prefer the relaxed notion, as it is more useful for tracking the voicings that are my primary analytical concern; it also produces more manageable groupings that, in my experience, are more analytically fruitful. This is because musicians often do consider B–D–F–A♭, C–E–G–B, and C–E–G♯–B to be tertian, just as they consider B–C–D♭ and C–D–E to be clusters.

Second method: Scalar embedding

The second approach is to think of a chord such as A–C–E–G♯ as an exact sequence of intervals inside some seven-note scale such as D acoustic (A melodic minor ascending). In that collection each interval is exactly two scale steps large; it is just that the scale steps themselves are somewhat uneven: **A** *b* **C** *d* **E** *f♯* **G♯**. The challenge is to find a group of scales that is large enough to contain a wide range of chromatic sets while also being similar enough to agree about their structure.

One such collection comprises the seven "Pressing scales," the largest possible equal-tempered sets without chromatic clusters. Four of these are the most even seven-note scales: diatonic, acoustic (melodic minor ascending), harmonic minor, and harmonic major (harmonic minor with raised third). One is the octatonic, the maximally even eight-note scale. The last two are hexachords: the completely even whole-tone scale and the less even but heavily triadic "hexatonic" scale formed from alternating semitones and augmented seconds. In my earlier work I explored these collections as harmonic objects in their own right (Tymoczko 2004, 2011); here I use them as "grids" for conceptualizing the

20 I do not count the "wraparound" interval from the top of the stack to the octave transposition of the bottom.

21 Mead 1997–98: 87 uses complete generic equivalence.

chromatic world. Thus, for example, we can define tetrachordal "clusters" as all those four-note chromatic sets that appear consecutively in one of these scales, including 0134 (which appears in the acoustic, octatonic, and harmonic scales), 0145 (harmonic and hexatonic), 0136 (harmonic), 0346 (harmonic), 0235 (diatonic, acoustic, harmonic, octatonic), and 0246 (diatonic and whole-tone). In this way the scalar set-class 0123 serves as a genus containing a variety of chromatic species (0134, 0145, 0346, 0235, and 0246).[22] This provides a notion of cluster that is both intuitive and expansive.

The main technical challenge here is multivalence. The pitch classes G, B♭, and D appear as a stack of scalar thirds in three diatonic scales (F, B♭, and E♭), two acoustic scales (C and B♭), two harmonic minor scales (D and G), and two harmonic major scales (D and E♭). But they also appear as a stack of scalar fourths in B harmonic minor.[23] Should we consider G minor to be both a triad (024_7) and a stack of fourths (014_7), or should we consider it a triad only? The first answer leads to messy categories with a substantial degree of overlap, while the second leads to cleaner categories at the cost of oversimplification. In this article I assign a chromatic species to its most popular scalar genus, unless it appears as two different scalar set-classes with approximately equal frequency.[24] Thus I assign 037 to scalar set-class 024_7 (since the triadic embedding outnumbers the fourth-chord embedding 9 to 1), but 014 to both scalar set-classes 013_7 and 012_7.[25] This is because there is general agreement about the status of the minor triad, whereas there is no such agreement about the status of the chromatic 014 trichord. I also prioritize the seven-note scales, considering octatonic, whole-tone, and hexatonic membership only when no seven-note embedding is available.

Though chunking and scalar embedding seem quite different, they are in fact closely related. In the first we measure chromatically but group intervals together, chunking or binning them into larger categories based on approximate size. In the second we let scales do the binning for us, measuring intervals along larger collections containing our chord. The two approaches converge because the seven-note Pressing scales typically put a single scale degree in each "slot" defined by our approximate chromatic interval categories. The first approach will be more familiar to atonal musicians accustomed to thinking chromatically, while the second will be more familiar to tonal musicians accustomed to a hierarchy of

22 Matthew Santa (1999, 2000), Christoph Neidhöfer (2005), and I (Tymoczko 2011: §4.8–10) have all used scalar set-classes to compare sets in different scales, but typically in contexts where the scales are clearly present on the musical surface; in this section, I use scalar set-classes to categorize chromatic objects even when the scales do not appear. This allows us to consider chromatic set-classes 0134, 0246, and 0235 all to be clusters, even in completely chromatic environments.

23 And D hexatonic, but I ignore that since it is challenging to compare set-classes across cardinalities.

24 *Approximately equal* means "within one." Suppose a chord appears as set-class *X* in *n* different scales (with *n* the maximum, considering all the different embeddings into all possible Pressing scales) and as set-class *Y* in *m* different scales (with *m* the second-largest number of embeddings). I count the chord as both *X* and *Y* if *m* is equal to either *n* or $n-1$; otherwise, I count it only as *X*.

25 The notes C–C♯–E appear as 013_7 in three scales (F♯ acoustic, A♭ harmonic major, and C♯ harmonic minor, all of which place a note between C♯ and E) and as 012_7 in two (F harmonic major and F harmonic minor, where there is no note between C♯ and E).

collections, chord-within-scale-within-aggregate. Their in-practice similarity is a boon for musicians who want to move between worlds.

Both systems provide categories with sharp boundaries, being discrete rather than continuous. In music, however, it is often useful to have some flexibility; for example, the tetrachord 0125 is not a stack of one- or two-semitone intervals, nor is there any Pressing scale containing these four notes, yet the chord seems more cluster-like than tertian. Even if we do not want to consider it an unqualified cluster, we might want some way of expressing the thought that it is *almost* a cluster, much as we might want to say that the French sixth is both nearly tertian and nearly quartal. My next approach provides flexible or "fuzzy" categories allowing us to express these thoughts.

Third method: Quantization

Here we categorize chords by proximity to equiheptatonic set-classes—the voice-leading distance, measured in continuous space, to the nearest proper subset of any scale dividing the octave into seven exactly even parts. I describe this as *equiheptatonic quantization*, a forbidding term that has the advantage of both concision and precision.[26] Conceptually, the approach extends the clustering technique to larger collections: clustering categorizes each step-interval separately, sending it to its nearest equiheptatonic analogue; equiheptatonic quantization instead sends whole chords to their nearest equiheptatonic analogues, using equiheptatonic set-classes as prototypes for categorizing twelve-tone equal-tempered sets.[27] The equiheptatonic collection is a *tertium quid* that connects the previous categorization systems: it contains the same interval categories as the chunking system (second, third, fourth, and their compounds) while also being very close to nearly even seven-note scales such as the diatonic, acoustic, and harmonic. Indeed, the diatonic scale is as close to equiheptatonic as it is possible to get in twelve-tone equal temperament.

Equiheptatonic quantization presents a number of technical complications. The first is that calculating distances generally requires a computer.[28] The second is the need to translate continuous distances-from-prototypes into binary judgment of categorial membership. The issue is that a chord can sometimes be slightly closer to one set-class than to another: the French sixth B–D♯–F–A is very close to the equiheptatonic fourth chord and just a bit farther away from the equiheptatonic tertian tetrachord. Should we assign it to one category or both

26 Quantization is explored in Yust 2015 and Tymoczko 2013.

27 Contrast Quinn 2006, where generic prototypes always belong to the scalar universe containing the specific chords.

28 If we use the Euclidean metric, then there is a four-step algorithm for computing the voice-leading distance between set-classes *X* and *Y*: put both sets in close position, transpose both so that their pitches sum to the same value *c*, compute the Euclidean distance between the two ordered lists of pitches, and repeat the procedure for each mode of set *Y*, moving its bottom note to the top and transposing the resulting notes so they sum to *c*. Fractional pitch classes are sometimes required to achieve the correct transposition.

categories, or create a new category intermediate between the two? In this article I choose a cutoff distance to determine category membership, with chords assigned to a category if they are at least that close to its prototype; thus, French sixth will be considered both tertian and quartal. I choose a second cutoff distance to define near membership; thus, I write 014^1 to indicate that the chromatic 014 trichord is close to being a cluster (category 1), even though it is not officially considered one.[29] Finally, when quantizing to the equiheptatonic scale I generally ignore multisets, considering only prototypes without pitch-class duplication; this is because I find it counterintuitive to think of, say, the chromatic set-class 012_{12} as a version of the equiheptatonic multiset 001_7.

It is slightly surprising that the three methods converge as well as they do. The big picture is clear enough: equiheptatonic quantization tends to agree with scalar embedding because the seven-note Pressing scales tend to divide the octave nearly evenly, and equiheptatonic quantization agrees with chromatic chunking because equiheptatonic intervals lie within the boundaries of the chromatic categories. That is, the equiheptatonic step is 1.71, between the semitone and major second; the equiheptatonic third is 3.43, between the minor and major third, and the fourth is 5.14, which is just above the equal-tempered fourth. But the convergence is closer than might be expected from these numbers alone. After all, one can stack equal-tempered semitones to produce chromatic chords distant from the equiheptatonic cluster (e.g., 01234_{12}). The surprise is that the other equal-tempered intervals do not lead to analogous divergences, in part because the equal-tempered fourth is very close to the equiheptatonic fourth and in part because note repetition limits the stacking of tritones, major thirds, and minor thirds. In other words, the convergence between the methods is the by-product of several unrelated mathematical factors.

Fourth method: Voicing

Voicing and the intrinsic scale provide yet another route to approximate set theory, largely consistent with those considered above. Fundamentally, intrinsic steps provide a unit of distance that is agnostic as to chord structure: absolutely any tetrachord can be voiced (2, 1, 2), just as any pentachord can be voiced in open position. What results is a topological perspective in which interval content is of secondary importance (Tymoczko 2020b). These intrinsic categories are sometimes too broad, much as standard set theory's categories are sometimes too narrow. Approximate set theory tries to find a middle ground by considering chords that are nearly evenly spaced when voiced cyclically; for instance, quartal tetrachords are nearly evenly spaced in the (2, 1, 2) voicing, and quartal pentachords are nearly evenly spaced in open position.

Mathematically, we reconceive the left-hand ranges in Figure 1.3 as averages rather than precise values. For example, the bottom row shows the cyclic voicings

29 I use 1.1 semitones as the primary distance and 1.3 semitones as the secondary distance.

for chords whose (exact) generator is greater than a major third and less than a tritone ($4 < g < 6$). In twelve-tone equal temperament the only possibility is the perfect fourth ($g=5$). We can, however, obtain nearly evenly spaced chords using near fourths, that is, fourths and tritones with perhaps the occasional major third. This gives us not just the purely quartal C–F–B♭–E♭ but also such nearly quartal sonorities as C–F♯–B–E and C–F–B–E: all three belong to the same approximate category in each of the three systems we have just considered, and all three are nearly evenly spaced in the (2, 1, 2) voicing.[30] In much the same way, tertian voicings bring out the cyclic structure of chords whose generator lies between minor and major third ($3 < g < 4$). Here there are no exact equal-tempered options, but we can combine major and minor thirds to produce nearly even voicings whose average interval size lies between 3 and 4 (e.g., pentachords like C–E–G–B–D). The surprise is that the ranges in the left column of Figure 1.3, which are determined mathematically as explained in the appendix, correspond both to intuitive terms such as *quartal* ($4 < g < 6$), *tertian* ($3 < g < 4$), and *clustered* ($g < 2$) and to the categories produced by the chunking, scalar, and quantization approaches. This convergence is the mathematical basis of our investigation. Practically speaking, it ensures that chords in the same category can be voiced similarly, no matter which approximation we use.

In fact, we can identify the specific voicings that bring out different intervallic qualia for chords in every approximate category. Figure 2.2 summarizes the situation. Each line corresponds to a chord size, with approximate chord categories, or genera, labeled in bold: clustered, tertian, quartal, and equipollent (discussed below). For each genus I list a number of voicings. The characteristic voicing, shown in bold, brings out the chord's intrinsic quality, voicing a clustered pitch-class set as a stack of seconds, a tertian pitch-class set as a stack of thirds, and so on. The figure also includes both gapped and smeared voicings of various sorts. For tetrachords and larger chords, interval cycles of one type can be voiced as gapped stacks of another type; thus, a clustered tetrachord can be voiced as a gapped stack of thirds. For hexachords the different chord types merge so that one and the same collection is equally clustered, tertian, and quartal. Much of this article is devoted to explaining these relationships.

3. Approximate analysis

To give a feel for the virtues of approximate thinking, Figure 3.1 annotates the opening fourteen measures of Schoenberg's op. 11/1. I interpret this music as largely concerned with approximate interval shapes. The opening six-note melody consists in a pair of "third-plus-step" gestures, harmonized in the left hand by gapped fourth and near-fourth stacks, most of which are third-plus-step trichords. In the fourth measure the right-hand verticalizes and

30 These chords not only have an average interval size of about 5 when in the (2, 1, 2) spacing but also a small variance; that is, each interval is close to the average.

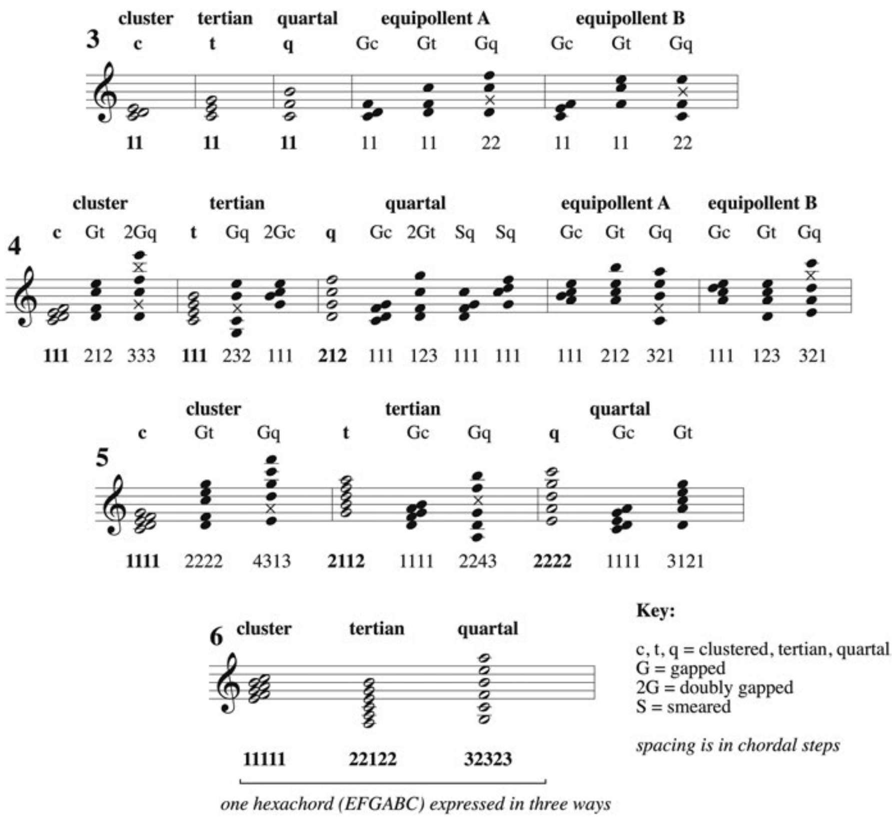

Figure 2.2. Some common voicings for fourteen genera: clustered, tertian, and quartal chords with three to six notes, along with equipollent trichords and tetrachords. Each line contains chords of a particular size. Above the staff I label chord categories and voicing types. The characteristic voicing is in bold and shown with open note heads; it creates a stack of approximately equal intervals. Additional voicings show how the chord can be represented as a gapped or smeared stack of some other category. Underneath each voicing is its pattern of spacing in intrinsic steps.

truncates the opening third-step melody. It is juxtaposed against a gapped third stack whose melodic intervals contract as the melody ascends, a common Schoenbergian strategy; these form the pattern third-third-step-step, or fifth-third-third-step-step if one includes the bass. This accompanimental figure also horizontalizes and fills in an open-position gapped-quartal trichord (cf. the open note heads in m. 6 of the example). The third iteration of the right-hand figure is harmonized in sevenths—a straightforward embellishment when we are thinking approximately, but not if we are looking for exact chromatic relationships.

When the melody reappears in mm. 9–10 it is subjected to the intricate but approximate algorithm shown in Figure 3.2: formerly we had a pair of third-steps joined by clusters (B–G♯–G and A–F–E, overlapping as clusters G♯–G–A and G–A–F); now we have third-step F♯–D–C followed by registrally inverted cluster

Figure 3.1. Approximate analysis of the opening of Schoenberg's op. 11/1.

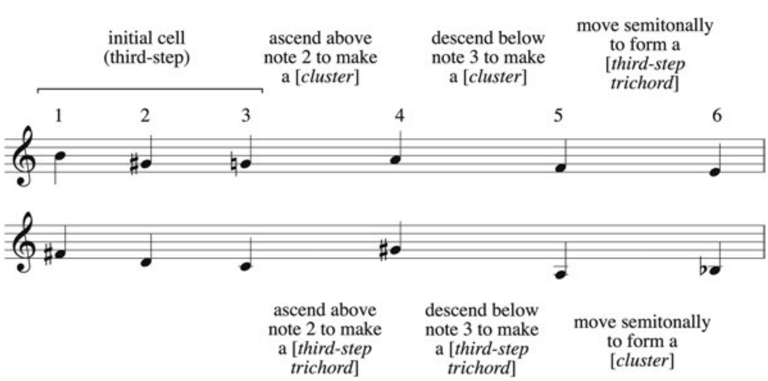

Figure 3.2. The two forms of the melody are produced by the same algorithm, interchanging the terms *third-step trichord* and *cluster*.

G♯–A–B♭, joined by registrally inverted third-steps (D–C–G♯ and C–G♯–A).[31] These subtle relationships are difficult to comprehend and even harder to hear. More palpable is the process of intervallic expansion from B–G♯–G through A–F–E to E–C–B♭ and F♯–D–C: minor third plus minor second, then major third plus minor second, then major third plus major second, moving from smallest to largest and omitting only minor third–major second. The left hand in mm. 12–14 uses sixths and steps, inverting the opening third-step configuration; the right hand ascends from fourths to an incomplete third stack, once again contracting as it rises; the contraction continues abstractly to seconds expanded registrally as ninths.[32] These ninths are echoed by sevenths and ninths in the right hand of m. 13.

Most of this is clear on the page and reasonably clear to the ear: if there is a perceptual challenge, it lies both in the speed of Schoenberg's gear changes and in his penchant for superimposing unrelated structures (e.g., sixths-and-steps against sevenths-and-ninths in m. 13).[33] It may be that Schoenberg went too far, but that would be a problem of execution rather than conception; I have little doubt that approximate organization is both intellectually coherent and perceptible in principle. Personally, I consider it sufficient in itself, not requiring supplementation by more rigid structures such as chromatic sets or twelve-tone rows.

Evidence of approximate thinking can be found throughout twentieth- and twenty-first-century music. Figure 3.3 shows two passages from the end of Ruth Crawford Seeger's nine Preludes for Piano, written in 1927–28 and published in 1941, one opposing right-hand sevenths with left-hand seconds, their approximate complements, and the other setting right-hand sevenths and quintic chords against quartal and quintic chords in the left. Figure 3.4 contains the opening pitches of Stockhausen's 1955 *Klavierstücke III*: all but two of the intervals are sevenths or ninths; the exceptions are gesture-initiating tritones. The approximate approach has also been discussed by numerous theorists: it features in Henry Cowell's ([1930] 1996: 111–16) *New Musical Resources*, a book sometimes credited with introducing harmonic clusters, and plays a central role in Vincent Persichetti's (1961: chaps. 3, 4, 6) harmony textbook.[34] There are at least three reasons why composers might favor approximate over exact organization. One is that it reflects a belief that music perception is approximate, with small intervallic variations not unsettling the listener's perception of musical similarity. Another is a desire for compositional options, and in particular for chromatic analogues to the

31 Haimo 1996 highlights the chromatic collections linking the melody's third-step trichords.

32 Perle 1962 identifies some of these relationships.

33 The superimposition of unrelated structures links set theory to polytonality; both are examples of the twentieth-century theme of *apartness*.

34 Ludmila Ulehla's (1966) *Contemporary Harmony* addresses clusters (224–29) and quartal and quintal harmony (chap. 17). Messiaen (1944) 1956: chap. 14 mentions quartal chords. More contemporary treatments include Sorce 1995: chap. 20 and Harrison 2016 (esp. 103–7, which distinguishes clusters from tertian harmonies, though it does not include a separate class of quartal harmonies). Many jazz sources discuss quartal harmony (e.g., Levine 1989: chap. 13).

Figure 3.3. (*top*) Measure 4 of Ruth Crawford Seeger's Prelude no. 9 balances sevenths in the right hand against seconds in the left, their approximate complements. (*bottom*) Measure 13 juxtaposes sevenths and quintic chords in the right hand against quartal and quintic chords in the left.

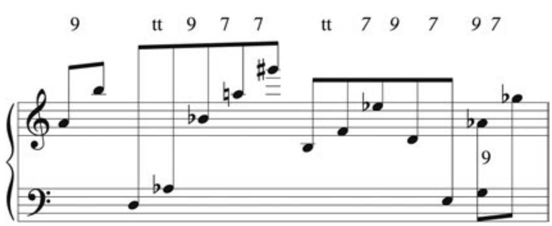

Figure 3.4. A reduction of the opening of Stockhausen's *Klavierstücke III*. Approximate intervals are labeled above the staff, using "tt" for tritone and 7 and 9 for sevenths and ninths. Italics represent minor sevenths and minor ninths; regular type, major sevenths and ninths.

subtle distortions produced by diatonic transposition. A third is a desire to categorize musical possibilities in a way that is responsive to their acoustic character.

What is fascinating is that approximate organization can be found even in twelve-tone music. Return to the opening of Schoenberg's violin concerto, shown in Figure 1.8. Andrew Mead (1997–98) has observed that the dyads in the passage, while not exactly related, are approximately equal: in each hexachord the solo violin states a minor second, which is answered by thirds and sevenths in the ensemble; in the first hexachord we have major third and minor seventh, while in the second we have minor third and major seventh. As Mead notes, the ensemble's melodic intervals also move by fourths and steps (first minor second and perfect fourth, then major second and tritone).[35] Approximate set theory thus reveals an additional layer of structure subsisting alongside the more rigid relationships of twelve-tone theory.[36] The rigid relationships could perhaps be compared to a grammar, structure provided by the language itself and present in any piece written in that language; approximate relations are more like semantics, statements a composer chooses to make within the constraints set by the grammar. Substantially more analytical effort has been directed toward the former than the latter, leaving us with a one-sided perspective on this multivalent music.[37]

At this point I should clarify that my interest in approximate interval size does not imply any disinterest in exact interval content. Two clusters, say, G4–G♯4–A4 and G4–A4–B4, can sound very different despite being clusters, and the same is true for tertian or quartal sonorities, even when voiced similarly. Musical similarity is multidimensional, and exact intervallic content is one of its dimensions. My argument, rather, is that approximate categories give us another dimension, particularly when reinforced by voicing: two clusters, such as C4–D4–E4 and A♭3–B♭3–C♭4, can sound chromatically different and yet generically similar. They are very different clusters, like *love* and *archaeopteryx* are very different nouns. I think the experience of hearing a pair of very different clusters is phenomenologically different from the experience of hearing totally unrelated chords (e.g., a consonant cluster and a dissonant stack of sixths). Approximate set theory captures one thread in the tapestry of relationships that jointly make up musical meaning.

That said, I do think there is an important philosophical question about the relative priority of exact and approximate. Allen Forte denied that terms such as *chromatic lines*, *thirds*, *triads* and *chords in fourths* could be usefully applied to Schoenberg's music (Forte 1972). Figure 3.5 shows two of Forte's analyses: in the first, he finds no fewer than eleven structurally significant tetrachords in a simple line of descending chromatic thirds; in the second he finds six structurally significant hexachords in the opening of Schoenberg's op. 11/1. Put aside questions about whether we can reliably segment notes in the appropriate way.

35 The two accompanimental tetrachords 0457 and 0356 are completely generically equivalent; in section 5 I place them in the "equipollent" category.

36 David Lewin (1998: 25–26) reported that Roger Sessions reported that Arnold Schoenberg conceived of the perfect fifth as "slightly more than half an octave." To me this suggests approximate categorization: the fourth is a half-octave or slightly less, the third is a quarter-octave or a little more, while the second is a sixth of an octave or less.

37 The comparison of twelve-tone rows to syntax echoes Edward T. Cone (1967), who likewise complains about analysts overemphasizing the syntactical at the expense of the expressive. Dubiel 1990 makes an analogous point about Babbitt. Haimo 1996 and Mead 1997–98 describe the role of approximate relationships in Schoenberg's music. Callender, Quinn, and Tymoczko 2008 and Tymoczko 2020b note that voice-leading geometry can be used to categorize chords but do not go beyond specific examples; the present article attempts to put that proposal on more systematic footing.

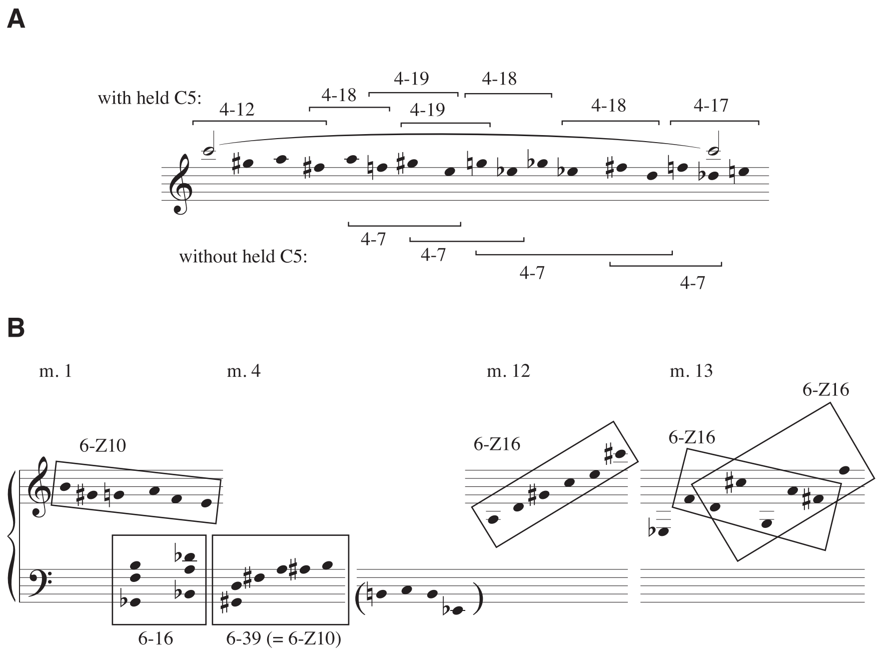

Figure 3.5. Two exact analyses by Allen Forte. A. Measure 17 of Schoenberg's *Herzgewächse* op. 17. B. The opening of Schoenberg's op. 11/1, analyzed in Figure 3.1.

Instead, consider just how demanding these analyses are: it is easy for me to imagine improvising or composing with approximate relationships, and I feel reasonably confident in my ability to identify them by ear; hearing or producing exact Fortean structures is much more difficult—and it is unclear that any aesthetic benefit is associated with success. (From what I can tell, listeners are not particularly sensitive to exact set-theoretical relationships.) The approximate analysis feels easy and natural, placing Schoenberg's music somewhere in the vicinity of free jazz; the exact alternative feels unrealistic and perverse, a kind of fantastical mathematics ungrounded in musical experience. Indeed, I would say something similar about the twelve-tone passage in Figure 1.8: to me, the approximate relationships feel like primary bearers of musical meaning, with the exact hexachordal relationship closer to being an intellectual conceit. The choice between exact and approximate has immense consequences for how we understand this music.

That choice is liable to be obscured by a line of reasoning both tempting and invalid, analogous perhaps to Kant's "transcendental illusions." The resemblance between approximate twelve-tone and exact seven-tone set theories can tempt the unwary musician into rejecting approximate set theory as an anachronistic residue of tonal thinking. In other words, we associate the terms *second*, *third*, and *fourth* with their exact diatonic meanings, not realizing that they can be given approximate chromatic meanings as well. Approximate set theory thus comes to seem contaminated with tonal cooties, deserving scorn and derision. Hence we find Forte rejecting such manifestly useful terms as *chromatic lines, triads,* and *chords in fourths*.

We can even reimagine diatonic or scalar set theory as a kind of approximation. Having absorbed the theoretical presumption of exactitude, I have always conceived scalar set theory as a brand of exact set theory, measuring in scalar rather than chromatic steps. This can be useful to be sure. But we can also interpret scalar set theory as dealing with the "hand shapes" of approximate set theory. It just so happens that, within the constraints of a seven-note scale, these hand shapes are typically realizable in only one way: given a starting note in some diatonic scale, the only way to create an approximate (chromatic) fourth chord is to use an (exact) diatonic fourth chord. The issue here is more metaphysical than technical, a question of the relative weight of exact and approximate: traditional theory postulates that listeners keep an accurate tally of all the intervals they hear, whether diatonic or chromatic; the alternative is that they imperfectly perceive approximate shapes, which sometimes get translated into exact relationships. We can reimagine the exact relationships as shadows on the cave wall: perhaps it is the approximate that is essential, with exactitude a mere appearance.

4. Trichords

I begin by categorizing trichords using approximate interval size, binning together (or "chunking") the six nonzero interval classes into three groups: small (seconds, one or two semitones), medium (thirds, three or four semitones), and large (fourths, five or six semitones), as well as their complements (sevenths, sixths, and fifths). We define *clusters* as chords that can be arranged as stacks of small intervals, *triads* as chords that that can be arranged as stacks of medium intervals, and *quartal chords* as stacks of large intervals. This produces four trichordal categories:

1. Clusters: 012, 013, 024;
2. Triads: 036, 037, 048;
3. Quartal: 027, 016;
4. Other (equipollent): 014, 015, 025, 026.

Each has a distinctive shape under the hand: clusters have their notes very close together, triads divide the octave nearly evenly, quartal chords have their notes clustered in two antipodal regions of pitch-class space, and the other/equipollent chords have two close notes with the third note at some distance—farther apart than the third note of a cluster, but closer than the third note of a quartal trichord. From this point of view, 014 is nearly clustered, while 026 is both nearly quartal and nearly tertian.[38]

38 The nearly tertian quality of the 026 trichord is exploited in R64 and R68 of Stravinsky's *Petrushka* and R89 of *Rite of Spring*, where the upper voices move in parallel major thirds while the lower voice moves so as to alternate between the minor triad and the 026 trichord; this is an example of Jonathan Russell's (2018) "kaleidoscopic oscillation."

Because I am biased toward thirds, I have always understood the "other" chords as gapped stacks of thirds. But we can also describe them as incomplete clusters or incomplete fourth chords: given the notes BCE, we can add D to form the cluster BCDE, G to form the tertian CEGB, or F to form the quartal CFBE. Thus BCE is equally balanced between the worlds of step, third, and fourth; I therefore say that it belongs to the *equipollent* category. The tertian hearing is of course culturally privileged, but let us put that aside while working out the abstract logic of the situation.

Summing up, we can say that every chromatic trichord either is a near interval cycle—cluster, triad, or quartal—or is equally balanced between these options, a "gapped" cycle missing one note, no matter which type of interval we choose.[39] Every cyclic chord has a characteristic voicing that brings out its cyclic structure. For trichords, the characteristic voicing is always close position (cf. Figure 1.3, where the close-position voicing is labeled CTQ, or Figure 2.2, where the characteristic voicings appear on the top staff). For these chords, the terms *cluster*, *tertian*, and *quartal* describe intrinsic qualia that can be reinforced by voicing but need not be, as they will sound clustered or tertian or quartal no matter how their notes appear in pitch. For equipollent chords, these same terms describe registral arrangements rather than intrinsic qualities. Suppose, for example, we want to emphasize that BCE is a near cluster; we can do this by voicing it in close position as a gapped stack of seconds, BC•E. If we instead want to emphasize its identity as an incomplete tertian sonority, we can voice it in close position as a gapped stack of thirds, CE•B. If we want to emphasize its quartal character, we should voice it in open position as a gapped stack of fourths, C•BE. Different voicings highlight different aspects of the set's interval structure.

Remember that only some of the chord's registral inversions bring out its cyclic quality: the voicings C4–D4–E4 and D4–E4–C5 are both in close position, but only the first is clustered. Looking back at the opening of Schoenberg's op. 11/1 (Figure 3.1), we see that the accompaniment presents a series of equipollent chords voiced as gapped fourth stacks, with a seventh separating the lower two notes; these gapped quartal voicings are preceded by an open-position 016 trichord, perhaps priming us to hear the later chords' quartal structure. It is also worth noting that the nearly quartal voicing of the 014 trichord, appearing as Schoenberg's B♭•AD♭, is anomalous within the equipollent group, its "quartal" voicing involving a four-semitone near fourth A–D♭. It is the only equipollent chord having this quality, residing near the boundary between equipollent and cluster.

All of this suggests a series of ear-training exercises. One could start with cyclic voicings of cyclic trichords, using pitch and register to reinforce pitch-class content. Having mastered this comparatively simple task, one might then proceed to "open" voicings in which adjacent notes are separated by fifths, sixths, and

39 My taxonomy of trichords is reasonably similar to those offered by Ernst Krenek and Paul Hindemith (see Harrison 2016: 49–57).

sevenths; these are harder to recognize as the ear needs to comprehend widely separated notes. One could then proceed to consider arbitrary spacings, emphasizing pitch-class content at the expense of voicing and pitch. But it would also be possible to go in the other direction, asking students to recognize the different equipollent voicings. Here pitch-class content is irrelevant, as every equipollent chord can be voiced as a gapped cluster (close position and spanning 4–6 semitones), as a gapped stack of thirds (close position and spanning 10–11 semitones), or as a gapped stack of fourths (open position, spanning 15–16 semitones). In this context adjectives like *quartal* and *clustered* refer only to spacing in pitch and not to intrinsic intervallic content. The ear-training task requires that we treat G3–F4–B♭4 as a gapped fourth stack rather than as a third stack, and D4–E4–G4 as a gapped stack of seconds.[40] This requires attending to features often consigned to the domain of taste and intuition.

If we categorize trichords by the nearest equiheptatonic set-class (ignoring multisets), we recover the same classification:

1. Clusters (equiheptatonic 012_7): 012, 013, 024;
2. Triads (equiheptatonic 024_7): 036, 037, 048;
3. Fourth chords (equiheptatonic 014_7): 027, 016;
4. Equipollent (equiheptatonic 013_7): 014^1, 015, 025, 026^3.

The superscripts indicate that some of these chords are very close to a template other than that to which they are officially assigned: the notation 014^1 indicates that the equipollent 014 almost belongs to category 1 (the clusters or one-step cycles), and 026^3 indicates that the equipollent 026 is close to being quartal (category 3, the three-step cycles).[41]

Practically speaking, there is no difference between chunking chromatic intervals and quantizing to the nearest equiheptatonic set-class. However, equiheptatonic quantization can help us understand deeper features of the underlying musical logic. In the seven-tone universe there are only four trichords: the cluster 012_7, the triad 024_7, the fourth chord 014_7, and the equipollent 013_7 (along with its inversion 023_7). The 013_7 set can easily be seen to be a gapped cluster **0**–**1**–[*2*]–$\mathbf{3}_7$, a gapped tertian chord **1**–**3**–[*5*]–$\mathbf{0}_7$, and a gapped fourth chord **1**–[*4*]–**0**–$\mathbf{3}_7$. This last arrangement spans more than an octave and hence requires an open-position voicing. The equipollent set-class is the only one that is not inversionally symmetrical, with its inversion 023_7 turning the preceding shapes upside down: **0**–[*1*]–**2**–$\mathbf{3}_7$, **3**–[*5*]–**0**–$\mathbf{2}_7$, and **0**–**3**–[*6*]–$\mathbf{2}_7$.

Equiheptatonic categories can help us track what might otherwise feel like an overwhelming abundance of musical possibilities. The top line in Figure 4.1 shows a sequence of voice-leading moves in diatonic space, here representing

40 Interested readers are invited to explore an ear-training website I have built (https://www.madmusicalscience.com/eartraining-voicing.html).

41 Subscripts and superscripts have very different meaning: subscripts identify the size of a containing scale, whereas superscripts identify near-membership in various approximate set-classes.

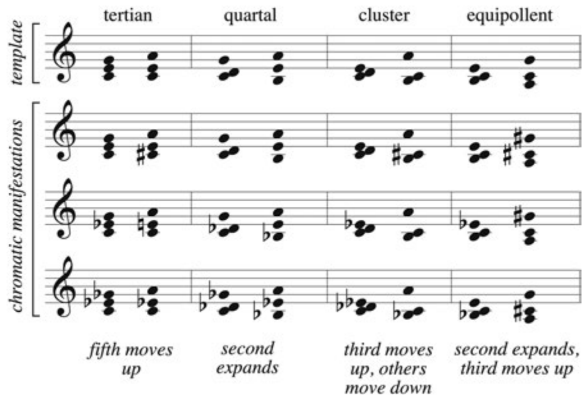

Figure 4.1. Chromatic voice-leading possibilities understood with reference to a diatonic (or equiheptatonic) template. In each column the chromatic voice leadings can be understood as manifestations of the diatonic template at the top. Beneath each column is a heuristic description of the underlying pattern.

the equiheptatonic scale. The voice leading in question is what I call the *basic voice leading*, a template that combines with transposition to generate all the spacing-preserving voice-leadings between transpositionally related trichords (Tymoczko 2020b). Underneath the example I show how the equiheptatonic pattern appears in chromatic space. These chromatic variants can be understood as manifestations of the same equiheptatonic template; though different in their details, they are broadly similar under the hand. The diatonic/equiheptatonic model can thus function as a mnemonic, a tool for internalizing a large number of chromatic options.

Now let us reconsider these phenomena from the point of view of concrete scale membership. For each of the twelve chromatic trichords we ask what scalar set-classes it forms when embedded in the seven-note Pressing scales (diatonic, acoustic, harmonic minor, and harmonic major):

1. Clusters (scalar 012_7): 013, 024, **014**;
2. Triads (scalar 024_7): 036, 037, 048;
3. Fourth chords (scalar 014_7): 027, 016;
4. Equipollent (scalar 013_7): **014**, 015, 025, 026;
5. None: 012.

There are two main differences from the previous lists: there is no Pressing scale containing the 012 chromatic cluster, and the 014 trichord is both cluster-like and equipollent.[42] (In lists of set classes, I use boldface to identify chords belonging to multiple categories.) This ambiguity recalls a fact mentioned earlier, that the

42 As discussed in section 2, I discard unusual embeddings, such as the fourth chord G–A♯–D.

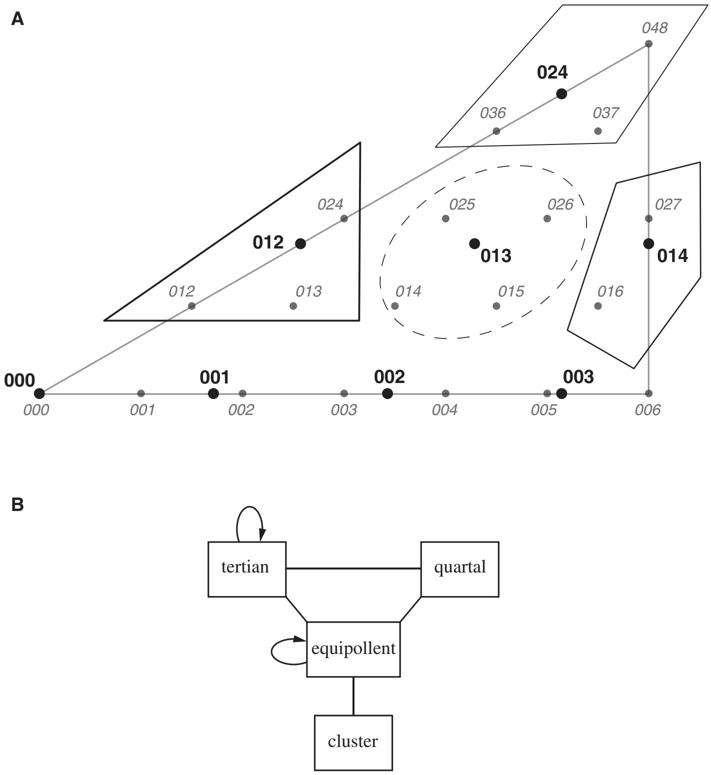

Figure 4.2. A. Three-note set-class space, with regions corresponding to clustered (triangle), tertian (quadrilateral), quartal (pentagon), and equipollent (oval) set-classes. Small italic labels show equal-tempered chromatic sets; large bold labels show equiheptatonic set-classes. B. Single-step voice leadings among equiheptatonic set-classes.

"incomplete quartal" voicing of the 014, **C**–[*f*]–**B**–**E♭**, requires a four-semitone "near fourth." In other words, the 014 trichord is the most clustered of the equipollent chords.

These ideas have a beautiful geometry. Figure 4.2A labels chromatic and equiheptatonic set-classes within the continuous space representing three-note set-classes. The space is bounded on left and right by the line of generated collections (interval cycles), which appears to change direction at the vertex of the triangle: beginning at the lower left with the triple unison 000; ascending along the left boundary as the size of the generating interval increases; passing through 012, 024, and 036 and then reaching the augmented triad at the top of the triangle; further increasing the size of the generator moves us downward along the right boundary until we reach the tritone with doubled note (060 or 006). Increasing the generator beyond that point retraces this same path in the opposite direction. (The bottom of the space contains multisets, which I ignore.) Figure 4.2A highlights regions containing the different chord categories, enclosing clusters in a triangle, triads in an quadrilateral, and quartal harmonies in a pentagon. Equipollent chords are in an oval at the center of the space, equally far from the boundaries. There are multiple chromatic chords in each region, gathering the isolated points of traditional set theory into larger communities—countries rather than city-states. There is also a single equiheptatonic collection in each region, serving as an abstract prototype and occupying a position remarkably close to the region's center. This is the geometrical link between the approximate twelve-tone and exact seven-tone worlds. Figure 4.2B presents a more abstract graph that shows how the approximate set-classes are related by single-step voice leading, with the equipollent chord in the center of the network.

Interval cycles are central to a range of chord-classification systems, dating back to Cowell, Persichetti, and Howard Hanson and continuing with the work of Forte, Quinn, Jason Yust, Richard S. Parks, and Robert D. Morris.[43] Cowell, Persichetti, and I are the only writers in this group who use coarse-grained generic intervals rather than fine-grained chromatic species (e.g., "second" instead of "major second"). In my view these approximate categories have many advantages. One is their close connection to compositional possibility: chords within a category often have similar capabilities (e.g., all the equipollent chords have an open-position voicing in which one pair of adjacent notes is separated by about five semitones while the other pair is separated by about twice that distance). They are also conducive to compositional flexibly, encouraging composers to use a range of closely related set-classes rather than just one (e.g., using both 027 and 016 rather than just 027). Perceptually, approximate categories leave room for listener imperfection, allowing that people might sometimes be able to hear that a chord is quartal without identifying its exact intervallic structure. Finally, they allow us to view similar phenomena from both chromatic and scalar perspectives, showing that the same procedures might be used in both tonal and atonal contexts—or in music combining tonal and atonal ideas.

And even though we are using approximate categories, we can still make precise analytical statements about their relationships. In previous work I have shown that any *n*-voice voice leading can be generated by repeatedly applying five elementary transformations: transposition along a chord, transposition along a scale, spacing-preserving neo-Riemannian transformations, normal-form-preserving perturbations, and pairwise voice exchanges (Figure 4.3). All but the voice exchanges preserve voicing, or spacing in chordal steps; all but the perturbations preserve set-class.[44] These same voice leadings can be found in the approximate domain as well—simple transformations complementing the simple chord categories of approximate set theory.

43 Cowell (1930) 1996; Hanson 1960; Persichetti 1961; Forte 1988; Parks 1989; Morris 1982, 1993, 1997; Quinn 2001, 2006, 2007; Yust 2016.

44 Unrestricted perturbations can be chained together to produce any bijective voice leading whatsoever; following John Roeder (1984, 1987), I limit them to those relating normal-form chords. In approximate set theory we often want to distinguish perturbations that preserve approximate set-class from those that do not.

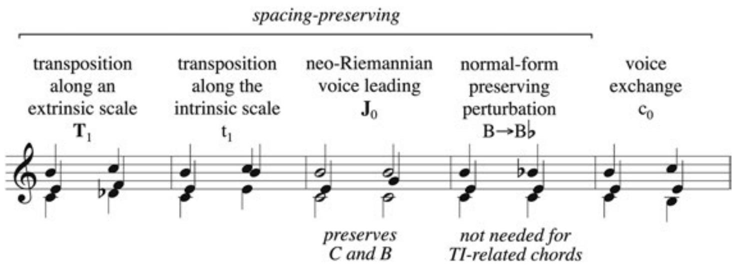

Figure 4.3. The five basic voice-leading transformations: transposition along an extrinsic scale, transposition along the intrinsic scale, neo-Riemannian voice leadings, normal-form preserving perturbations (not needed when chords are TI-related), and the small voice exchange c_0 that swaps the chord's closest notes, moving them in exact contrary motion by the smallest possible distance.

Figure 4.4 uses these transformations to analyze the voice leadings connecting the accompanimental trichords in the opening of Schoenberg's op. 11/1. All but chord 7a are in open position with a seventh between the lower two voices; all but chords 5 and 7a used the gapped quartal voicing on Figure 2.2. Most are equipollent 013_7 trichords, though there are two "near equipollents" a semitone away. (To treat these as near equipollents is to recognize the fuzziness of the boundaries between approximate set-classes.)[45] At each change of chord there is a single-step perturbation that slightly alters the underlying set-class. The first three voice leadings are what Joseph Straus (2003) describes as "nearly transpositional," moving voices by nearly the same amount. The fifth chord is an 023_7 trichord, connected to its neighbors by the neo-Riemannian voice leading preserving the lower-voice seventh ($\mathbf{J}_0$).[46] The final voicing is in close position, with the melody's B♭ supplying the expected tenth above the bass. While this analysis is technical and precise, the underlying relationships are quite intuitive, simple hand shapes transformed in simple ways: open-position trichords voiced with a tenth in the bass. The surprise is that intuitive exploration gives rise to such a complex yet surveyable landscape.

As a modernist icon, Schoenberg's op. 11/1 is rivaled only by Igor Stravinsky's *Rite of Spring*—a piece that also begins with open-position equipollent trichords

45 Schoenberg uses every approximate set-class containing a step, which is to say, every approximate set-class other than tertian.

46 As in section 1 above, I use *neo-Riemannian voice leading* to refer to spacing-preserving voice leadings from a chord to its inversion. The voice-leading $\mathbf{J}_0$ is the neo-Riemannian voice leading preserving the notes of a pitch-class set's smallest interval. For a consonant triad, this is the neo-Riemannian L transformation. The voice leading $\mathbf{J}_{2i}$ preserves the pair of notes *i* steps below the smallest interval in the inversional normal ordering, considered as circular. For a consonant triad, $\mathbf{J}_1$ is R and $\mathbf{J}_2$ is P. For odd cardinality, this defines all the operations $\mathbf{J}_i$. For even cardinality, the odd inversions $\mathbf{J}_{2i+1}$ preserve the notes of the two-step intrinsic interval between the upper note of the intrinsic step preserved by $\mathbf{J}_{2i}$ and the lower note of the intrinsic step preserved by $\mathbf{J}_{2(i+1)}$. So, for the chord A–B–D–F, $\mathbf{J}_0$ preserves (A, B), $\mathbf{J}_1$ preserves (F, B), $\mathbf{J}_2$ preserves (F, A), and $\mathbf{J}_3$ preserves (D, A). For a chord with multiple smallest intervals, different analytical circumstances may suggest different choices about which voice-leading to label $\mathbf{J}_0$. These definitions derive from the definition of i_x in Tymoczko 2020b: 32–33. Readers can explore these transformations using my online voice-leading calculator at https://www.madmusicalscience.com/nr.html.

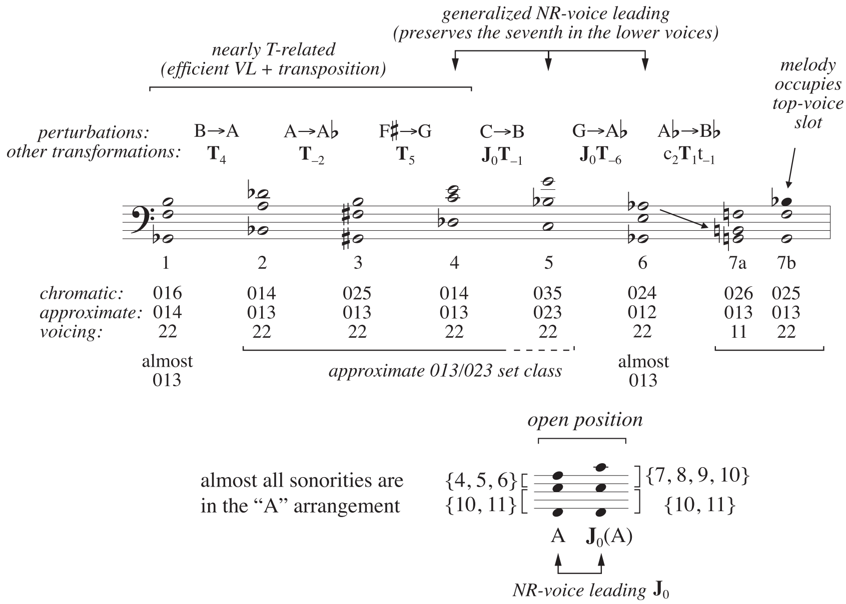

Figure 4.4. Voice leadings at the opening of Schoenberg's op. 11/1. To obtain the progression, apply the perturbation and then the transformation. For example, starting with the first chord, move B to A and then transpose up by four semitones. Almost all the trichords are in the "A" arrangement, with a seventh between the lower two voices (10 or 11 semitones) and a major third or fourth between the upper two (4, 5, or 6 semitones); the fifth chord is generated from the neo-Riemannian (NR) inversion J_0, preserving the lower-voice seventh and moving the top voice so that it forms approximate set-class 023, the inversion of 013.

with a fixed distance between their lower two voices. Figure 4.5 shows that the fixed distance is a fourth and that all the chords belong to the same approximate set-class. The passage uses just three transformations: the single-semitone displacement exchanging 045 and 035 trichords (or their inversions, 015 and 025), the neo-Riemannian voice leading $\mathbf{J}_2$ preserving the lower-voice fourth, and chromatic transposition.[47] All of the approximate 023 chords are voiced as gapped fourth stacks, giving the passage a quartal quality.[48] It is intrinsic to the musical logic that the approximate 013 trichords are not gapped fourth stacks, for if one wants open-position voicings between approximately inversionally related trichords, then one has to choose between lower-voice parallelism and the gapped quartal spacing. Figure 4.6 includes two recompositions illustrating this dependency. Both use the same melody, the same open-position voicing, and the same sequence of exact and approximate set-classes. The first shifts the parallelism to the outer voices so that they move in approximate tenths; this constraint,

47 This analysis in terms of approximate inversion is due to Russell 2018.

48 Stravinsky's fourth + seventh arrangement is the inversion of the more common seventh + fourth, appearing in Schoenberg's op. 11/1.

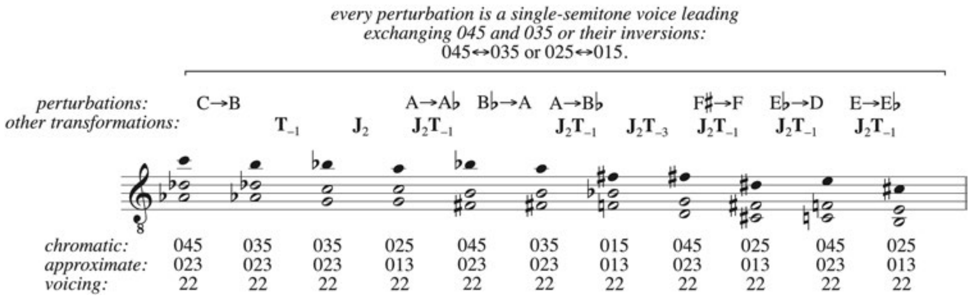

Figure 4.5. The opening of Stravinsky's *Rite of Spring* presents a series of equipollent trichords. The neo-Riemannian J_2 transform preserves the perfect fourth. Once again, apply the perturbation before the other transformations.

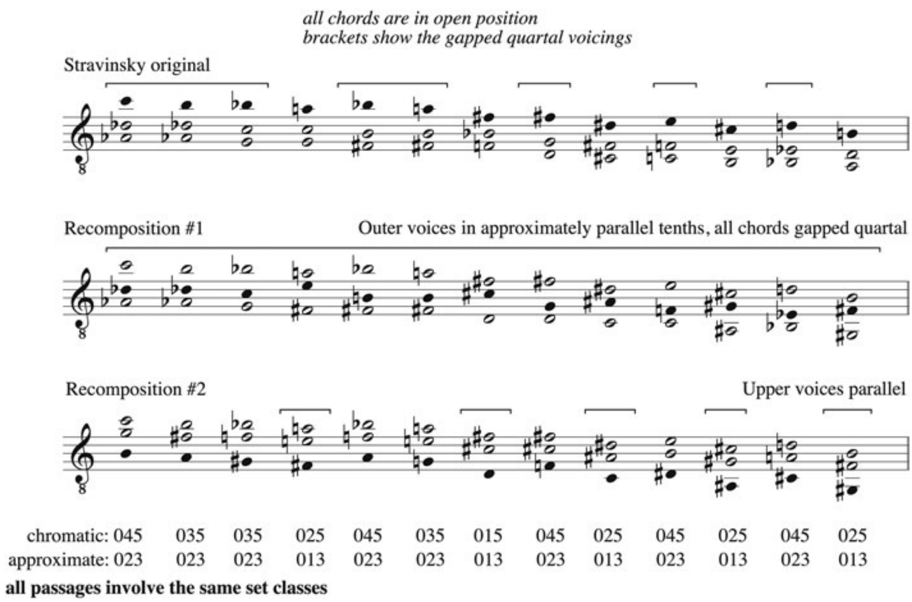

Figure 4.6. A reduction of mm. 4–6 of *Rite of Spring*, along with two recompositions that shift the parallel motion to other pairs of voices. This moves the position of the gapped quartal voicings.

along with the open voicing and the shared set-class structure, ensures that every chord is a gapped fourth-stack.[49] The second shifts the parallel fourths to the upper voices, moving the gapped quartal voicings to the approximate 013 sets, exactly where the original passage lacked them. I think the first recomposition sounds more quartal but less parallel than Stravinsky's original, as all the harmo-

49 The preservation of exact set-class forces the parallelism to be approximate rather than exact, since 045 and 035 have different species of third.

nies are voiced as gapped fourth stacks, but the parallelism is only approximate. The second recomposition has fewer gapped quartal voicings, but the parallelism is clearer. Here approximate set theory reveals the complex interrelationships among the different features of Stravinsky's phrase.

5. Tetrachords

Tetrachords offer a substantially more complex array of compositional possibilities. Once again, we begin by "chunking" intervals, considering clusters to be stacks of seconds, tertian chords to be stacks of thirds, and quartal chords to be stacks of fourths:

1. Clusters (6): 0123, 0124, 0134, 0135, 0235, 0246;
2. Tertian (5): 0148, 0158, 0258^{3}, 0358^{3}, 0369;
3. Quartal (4): 0156, 0157, 0167, 0257^{1};
4. Equipollent (4): 0137, 0237, 0136^{1}, 0247^{13};
5. Noncyclic (10): 0126, 0127, 0347, 0125^{1}, 0145^{1}, 0236^{1}, 0146^{13}, 0147^{3}, 0248^{2}, 0268^{23}.

Superscripts again indicate proximity to a second category: superscript 1 indicates a near cluster with a single three-semitone step, superscript 2 indicates a near-tertian chord with a single diminished third, and superscript 3 indicates a near fourth chord with a single diminished fourth. Equipollence is defined intervallically: when using clustering, an equipollent tetrachord is contained within a five-note cluster, tertian, and fourth chord.[50]

Each category has a distinctive shape under the hand: clusters have their notes as close together as possible, tertian chords divide the octave nearly evenly, quartal chords have two pairs of notes separated by roughly half an octave, and equipollent chords are grouped "3+1," with three notes close together (but not too close) and an outlier about two steps away from the rest. Knowing the shapes allows you to immediately recognize a chord's compositional potential.

Figure 5.1 shows the pitch-class content of a four-note interval cycle as the size of the generating interval *g* gradually increases: when *g* is four semitones or less, the interval cycle spans less than an octave; when *g* is between four and six, the top note lies between the octave transposition of the first and second notes, producing the (2, 1, 2) voicing. Thus the characteristic voicing is close position for clusters and tertian chords, and (2, 1, 2) or "drop 2" for fourth chords. As before, the equipollent chords use different voicings to express their different affiliations: to represent ABCE as an incomplete stack of seconds, we use close position (ABC•E); to represent it as an incomplete stack of thirds, we use the (2, 1, 2) voicing (ACE•B, or **A** *b* **C E** *a* **B**); and to represent it as an incomplete stack of fourths, we use the (3, 2, 1) voicing (C•BEA or **C** *e a* **B** *c* **E A**). The equipollent

50 In general, we can define an *n*-note chord as equipollent if it is contained within an (*n*+1)–note cluster, stack of thirds, and stack of fourths.

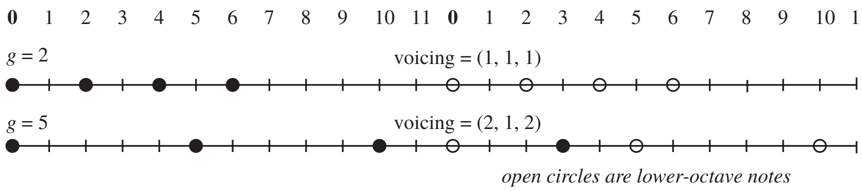

Figure 5.1. Generated tetrachords with generating intervals $g = 2$ and 5.

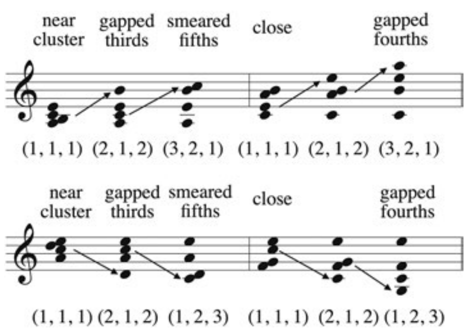

Figure 5.2. Octave displacements relating the close, tertian, and (3, 2, 1) or (1, 2, 3) voicings. Either the second lowest note moves up by an octave or the second highest note moves down by an octave.

FABC, the inversion of ABCE, instead uses the (1, 2, 3) voicing to express its near-quartal aspect (CFB•A or **C F** *a* **B** *c f* **A**).

These voicings are systematically related by a series of octave displacements: to go from close position to (2, 1, 2) we can move the second lowest note up by an octave; to go from (2, 1, 2) to (3, 2, 1) we can repeat the procedure, moving the second lowest note up by an octave once again (Figure 5.2).[51] Figure 5.3 graphs this process: moving between concentric squares changes the voicing, while moving along the squares transposes along the intrinsic scale, changing registral inversion. Though the graphs look simple on the page, it takes considerable effort to internalize them: interested readers might practice moving between these voicings on their instrument, changing set-classes and adding extrinsic transposition as they get more comfortable.

Arranging the cycles in the circular order cluster → tertian → quartal → (cluster) reveals an interesting symmetry: each approximate set-class is a singly gapped stack of its successor's intervals and doubly gapped stack of its predecessor's intervals. Thus the cluster is a singly gapped third stack, DF•CE voiced (2, 1, 2) in intrinsic steps, and a doubly gapped fourth stack, E•D•CF voiced (3, 3, 3). The tertian chord is a singly gapped fourth stack, GC•BE voiced (2, 3, 2), and

51 These transformations generalize guitarists' *drop-2* and *drop-3* nomenclature. Alternatively, as noted in Bicket 2001, one can obtain the (2, 1, 2) voicing (which pianist Barry Harris called "long") by crossing the outer notes of a close-position voicing (a "short chord" in Harris's terminology).

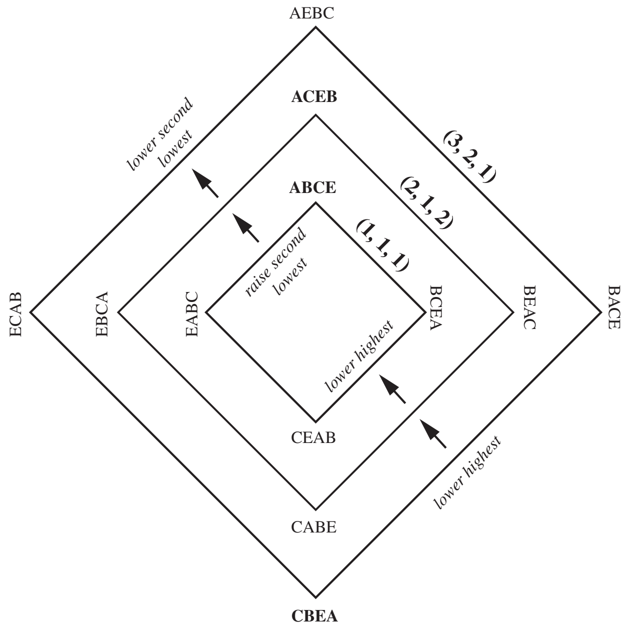

Figure 5.3A. A transformational graph relating voicings of ABCE. Boldface shows the gapped clustered, gapped tertian, and gapped quartal voicings.

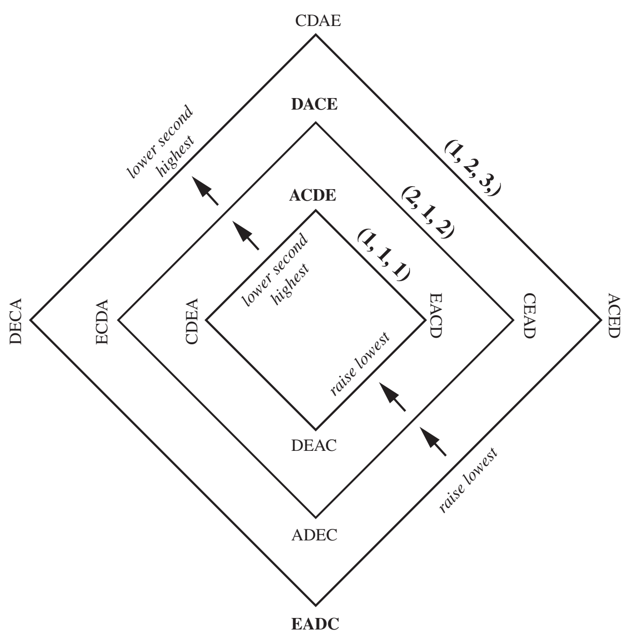

Figure 5.3B. A transformational graph relating voicings of ACDE. Boldface shows the gapped clustered, gapped tertian, and gapped quartal voicings.

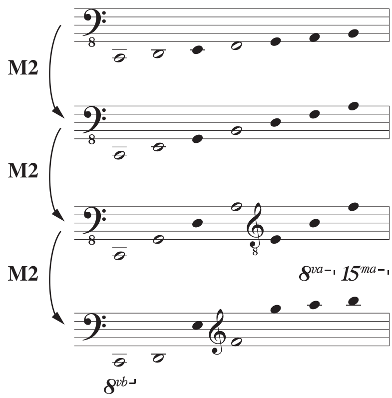

Figure 5.4. In a seven-tone scale, the M2 transform sends clusters to tertian, tertian sets to quartal, and quartal to cluster. The equipollent set-classes (open note heads = trichord, closed note heads = tetrachord) are preserved under the transform.

a doubly gapped cluster, G•BC•E in close position. And the fourth chord is a gapped cluster in close position, CD•FG, and a doubly gapped third stack, DF•C•G (1, 2, 3). These voicings are shown on the second staff of Figure 2.2.

This symmetry can be understood with reference to diatonic quantization. John Clough (1979, 1994) and Jason Yust (2009) have noted that, in any seven-note scale, the M2 transform sends steps to thirds, thirds to fifths, and fifths to ninths, which are steps when we ignore octave; indeed, a sequence of three successive M2 transforms preserves pitch class (Figure 5.4).[52] This explains the relations among the different approximate set-classes: a cluster, or gapped third stack, becomes a tertian chord when its intervals are multiplied by 2, and the cluster's "gapped tertian" voicing becomes "gapped quintal" (which per the appendix can be converted into gapped quartal). Figure 5.4 shows that the equipollent set-classes are invariant under this M2 transform, which in turn explains their status as gapped stacks of each interval.[53]

Once again, we can imagine a sequence of ear-training exercises, starting with the cyclic tetrachords voiced cyclically and progressing to the different voicings of the equipollent tetrachords. Here there is the added wrinkle that a tetrachord's intrinsic interval structure can conflict with its pitch-space arrangement. Intrinsically, the tetrachord C–D–E–F is clustered and not at all tertian, yet it can be arranged in register as an incomplete third stack, D4–F4–C5–E5. To my ear this arrangement sounds like a ninth chord. Something similar could be said for E3–B3–C♯5–G♯5, which sounds very much like a stack of fifths (missing only F♯4) despite having the abstract pitch-class content of a minor-seventh chord. Once again we see how chord quality depends on compositional choice: a student who treats D4–F4–C5–E5 as a cluster or E3–B3–C♯5–G♯5 as a minor-seventh chord, full stop, will likely be missing something. Pitch class does not always trump pitch.

52 Thanks here to Jason Yust.

53 The equipollent trichord C–D–F–(C) has step intervals 1–2–4, which are circularly permuted by M2, becoming 2–4–1 and 4–1–2. In other words, the trichord's internal intervallic structure mirrors the orbit of the transformational cycle.

There are also tetrachords without manifest cyclic structure. The majority of these are very nearly cyclic and can be assigned to the appropriate categories; indeed, both the 0268 French sixth and the 0248 "dominant seventh sharp five" are often categorized as nearly tertian. The remaining noncyclic sets can be understood as gapped but not equipollent; 0127, for example, can be expressed as a singly gapped stack of fourths 271•0 but not as a singly gapped stack of seconds or thirds; in this respect it is more quartal than tertian or clustered. One way to put this is that, while we can still voice these chords in the characteristic ways—as a cluster, stack of thirds, stack of fourths, or the various "equipollent" voicings—none of these voicings will be markedly even; these chords are more intervallically heterogeneous than those in the first four categories. What is surprising is not that such chords exist but that they comprise such a small part of the tetrachordal universe.[54]

Turning to concrete scale membership we obtain very similar categories:

1. Clusters (6, scalar 0123_7): 0134, 0135, 0145, 0235, **0236**, 0246;
2. Stacks of thirds (5, scalar 0246_7): 0148, 0158, 0258, 0358, 0369;
3. Fourth chords (6, scalar 0134_7): 0146, **0147**, 0156, 0157, 0257, 0268;
4. Equipollent (8, scalar 0124_7): 0136, 0137, **0147**, **0236**, 0237, 0247, 0248, 0347;
5. Octatonic (1, scalar 0145_8): 0167;
6. None (5): 0123, 0124, 0125, 0126, 0127.

The two systems agree completely about the tertian category and about the majority of the other cases. Many disagreements reflect minor differences of emphasis; 0146, for example, is a scalar fourth chord, but a near fourth chord in the chunking approach.

Categorizing chords by equiheptatonic quantization gives the following categories:

1. Clusters (7, equiheptatonic 0123_7): 0124, 0134, 0135, 0145, 0235, **0236**, **0246**^{3};
2. Stacks of thirds (7, equiheptatonic 0135_7): 0148, 0158, 0369, 0358^{34}, **0268**, **0258**^{4}, **0248**;
3. Stacks of fourths (7, equiheptatonic 0134_7): 0156, 0146^{14}, 0157^{2}, 0257^{4}, **0268**, **0258**^{4}, **0147**^{2};
4. Equipollent (10, equiheptatonic 0124_7): 0126, 0137, 0237, 0136^{1}, 0347^{1}, 0247^{23}, **0236**, **0246**^{3}, **0248**, **0147**^{2};
5. None (4): 0123, 0125^{1}, 0167^{3}, 0127^{4}.

This is a similar list with the same basic hand shapes: clusters, triads, quartal chords grouped 2+2, and equipollent chords grouped 3+1.[55] However, the categories overlap more than in the other systems. Once again, equiheptatonic

54 Some of these noncyclic sets are quite close to equiheptatonic multisets, which I discard in my categorization. Hanson 1960 has an analogue to my noncyclic category.

55 Once again, boldface is used for chord duplication, and superscripts are used to indicate proximity to some category other than the one to which a chord is assigned thus 0146, though closest to the equiheptatonic fourth chord, is fairly close to the cluster (with just one interval of size 3) and fairly close to the equipollent category.

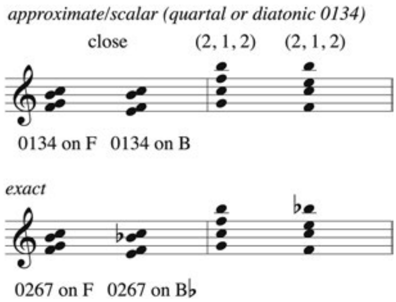

Figure 5.5. Efficient descending-fifth voice leading connecting quartal voicings, in both diatonic and chromatic space.

categories can help us understand voice leading; for example, a descending-fifth progression of equiheptatonic fourth chords can be voiced with pairs of voices descending in alternating steps, and similar patterns are available in twelve-tone equal-temperament (Figure 5.5).

Figure 5.6 compares the different systems, aligning tetrachords spatially. I do not plan to adjudicate among them, as I am more impressed by their convergence than by their differences. In each case there are four basic chord shapes: clusters and tertian tetrachords, whose cyclic structure is manifest in close position, fourth chords grouped 2+2, and "equipollent" chords grouped 3+1, plus a few outliers. An atonal composer might think about these shapes using approximate interval sizes, a tonal musician might think about how they sit inside familiar scales, and a computationally savvy music theorist might derive them via equiheptatonic quantization. Different musicians may also draw slightly different boundaries—one considering 0146 a near cluster, another considering it a near fourth chord. The details are less important than the big picture.

That picture again has an elegant geometry, though it is more difficult to visualize since four-note set-class space is inherently three-dimensional. Figure 5.7 graphs a two-dimensional cross section of the space, containing all tetrachords whose smallest interval is a semitone; superimposed on it is the one-step layer of equiheptatonic space, containing all the equiheptatonic tetrachords without doublings. (These equiheptatonic set-classes lie in the third dimension, a centimeter or so above the paper.) The line of generated tetrachords passes three times through the semitonal chromatic layer at the three points marked with circles: *0123* (which also appears on the figure as *012B*), 0, 3.66, 7.33, 11 (which is on the boundary in the **0135** region), and 0, 5.5, 11, 4.5 (on the boundary in the **0134** region). Each of these intersections corresponds to a cyclic category: the triangular region contains clusters, the quadrilateral contains third stacks, and the pentagon contains fourth stacks. These are again contiguous regions in the space, though now with some overlaps and with a few chords outside each region. Equiheptatonic prototypes are very close to the center of the regions they represent.

	chunking	concrete scale	7tet
1. clusters	0123, 0124, 0134, 0135, 0235, 0246	0134, 0135, 0145, 0235, 0236, 0246	0124, 0134, 0135, 0145, 0235, 0236, 0246^{3}
2. tertian	0148, 0158, 0258^{3},0259^{3}, 0369	0148, 0158, 0258, 0259, 0369	0148, 0158, 0258^{4},0259^{34}, 0369, 0268, 0248
3. quartal	0156, 0157, 0167, 0257^{1}	0146, 0147, 0156, 0157, *0167*, 0257, 0268	0146^{14}, 0147^{2}, 0156, 0157^{2}, 0257^{4}, 0268, 0258^{4}
4. equipollent	0136^{1}, 0137, 0237, 0247^{13}	0136, 0137, 0237, 0247, 0147, 0236, 0248, 0347	0136^{1}, 0137, 0237, 0247^{23}, 0147^{2}, 0236, 0248, 0347^{1} 0246^{3}, 0126
noncyclic	0125^{1}, 0126, 0127, 0347, 0145^{1}, 0236^{1}, 0146^{13}, 0147^{3}, 0248^{2}, 0268^{23}	0123, 0124, 0125, 0126, 0127	0123, 0125^{1}, 0167^{3}, 0127^{4}

Figure 5.6. Comparing the three different classification systems: chunking, concrete scale, and equiheptatonic (7tet). Tetrachords are aligned spatially within each row.

Analytically, this perspective helps us conceptualize relationships that might otherwise be difficult to describe. Schoenberg's *Pierrot Lunaire* opens with a heavy emphasis on thirds (Figure 5.8): the piano alternates between a tertian trichord and an equipollent tetrachord voiced as a gapped third stack, while the violin and flute combine to form a smeared triad.[56] The two tetrachords are members of the same set-class, but I am not certain that is important: given how many sets can be found in this music, and the paucity of systematic relationships, I find it more satisfying to listen approximately. Not only are the approximate relationships easier to hear, but they are more reliably present—indeed, they are characteristic of Schoenbergian atonality.

Another example comes from Hans Stuckenschmidt (1965), who observed that the progression in Figure 5.9 appears in the work of both Alban Berg and

56 The 0147/0367 tetrachord is equipollent in the concrete scale and equiheptatonic approaches but not in the chunking approach.

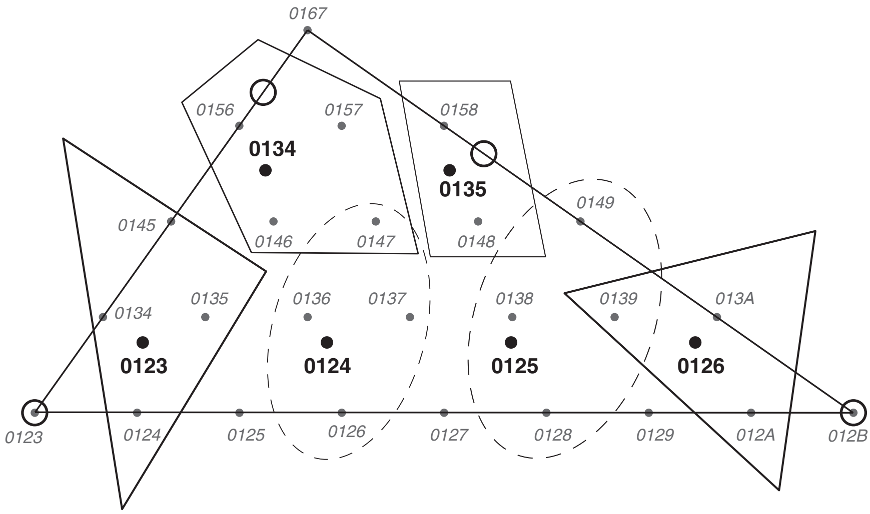

Figure 5.7. The single-semitone layer of chromatic tetrachordal set-class space, superimposing the one-step layer of diatonic tetrachordal set-class space. Circles represent generated collections. Shapes represent clustered (triangle), tertian (quadrilateral), quartal (pentagon), and equipollent (oval) set-classes. Small italic labels show equal-tempered chromatic sets; large bold labels show equiheptatonic set-classes.

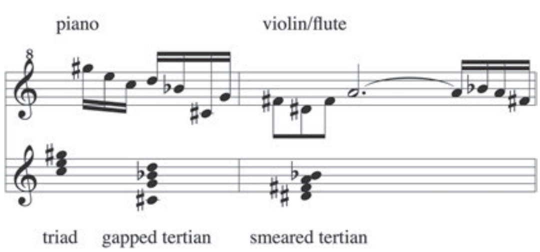

Figure 5.8. Tertian structures at the start of Schoenberg's *Pierrot Lunaire*.

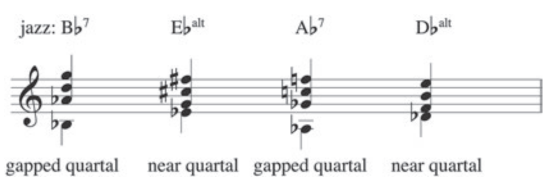

Figure 5.9. A progression used by both Debussy and Berg.

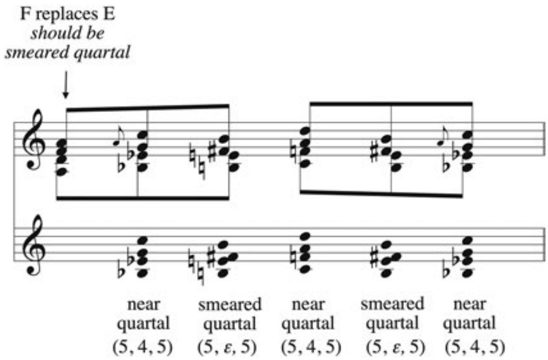

Figure 5.10. Measure 32 of *Rite of Spring*.

Claude Debussy. In Debussy one can interpret it as a distorted descending-fifth sequence alternating diatonic and altered-scale dominants; indeed, the progression is idiomatic in jazz and related genres. The more interesting question is how it functions in the nonscalar and almost atonal idiom of Berg's op. 2. Here it is useful to attend to the superimposed melodic intervals, perfect fourths in the bass against semitones in the upper voices, combining to produce somewhat quartal harmonies: a gapped fourth chord B♭•A♭DG alternating with the nearly quartal E♭GCF♯. One might think of the melodic intervals as rigid structures analogous to the twelve-tone ordering in Figure 1.8; the near-quartal verticalities are a resultant structure analogous to the approximate relationships discussed at the end of section 3. Approximate set theory thus helps us recognize similar patterns of thought appearing across the tonal/atonal boundary—not just in Berg's tonality and Schoenberg's twelve-tone music, but in Debussy as well.

A third example is the mysterious passage near the start of *The Rite of Spring* (Figure 5.10): a lower-register fourth moves in regular waves while an upper-register fourth alternately forms a smeared fourth stack and a nearly quartal voicing, (0, 5, 7, 12) and (0, 5, 9, 2) in semitones. I think pitch is as important as pitch class here: it is not just that the Stravinsky deploys two kinds of tetrachord each containing a pair of perfect fifths, 0057 and 0257, but that he voices the chords to highlight the intervallic resemblance.[57] Pitch and pitch class together create the musical effect.

Such passages reinforce the claim that nontonal music combines two different kinds of structure: a rigid syntax involving exact pitch-class relationships (sets, rows, etc.), and a more flexible system of approximate relationships, often manifested in pitch. The rigid structure underwrites the comparatively high-prestige discipline of posttonal theory and has dominated the discourse surrounding this music. The more flexible structure has been relegated to the less prestigious

57 This is very similar to Figure 4.5 and one of many passages in the *Rite* where one group of voices move in parallel while the remaining voices alternate to create two different set-classes, a technique Russell (2018) has called "kaleidoscopic oscillation."

domain of pedagogy and "composerly intuition"—and sometimes dismissed altogether.[58] The concept of voicing, understood by way of the intrinsic scale, allows us to make precise observations about this more flexible realm: the passages we have considered depend not just on pitch-class content, but on how pitch classes are placed in register. In many cases these pitch structures are immune to small perturbations, allowing us to identify commonalities invisible to standard post-tonal theory. Approximate listening thus provides a middle ground between set-theoretical exactitude and a very general gestural listening that disregards details in favor of broad trajectories of register, contour, and texture.[59]

6. Pentachords

Equiheptatonic quantization provides the simplest view of the pentachordal universe:

1. Clusters (9, equiheptatonic 01234$_7$): 01246, 01346, 01356, 01457, 02357, 01347^{2}, 01357^{2}, **02458**, **02468**^{3};
2. Tertian (9, equiheptatonic 01235$_7$): 01248, 01348, 01358, 01458, 02358^{1}, 02469^{3}, **02458**, **02468**^{3}, **01369**;
3. Quartal (9, equiheptatonic 01245$_7$): 01268, 01468, 01478, 01568, 02479, 01368^{2}, 01469^{2}, 02368^{12}, **01369**;
4. Noncyclic (14): 01234, 01235, 01236, 01237, 01245, 01256^{1}, 01257^{1}, 02346^{1}, 02347^{1}, 03458^{1}, 01367^{13}, 01247^{2}, 01258^{2}, 01267^{3}.

Clusters have their notes close together; tertian sonorities are arranged 4+1, with four notes close (but not too close) and one farther apart; and quartal chords are divided 3+2 into antipodal groups. In section 2 I showed that each cyclic category has its own characteristic voicing: close position for clusters, (2, 1, 1, 2) for thirds, and open position for quartal chords.

These shapes provide a quick way to recognize pentachordal affordances. Suppose you randomly plunk your fingers down on the piano and come up with the notes D–E–F–A–B♭. The nearly-even 3+2 pattern alerts you to the presence of quartal structure in open position: starting on F, the top note of the group of three, gives F–B♭–E–A–D: three perfect fourths and one "near fourth" or tritone. Other registral inversions are not quite so quartal: starting on D produces D–F–B♭–E–A, with a minor third replacing a perfect fourth. Conversely, the absence of 4+1 structure indicates that the chord is not a stack of thirds, and

58 This is not just a conflict between theory and practice but a tension within practice, for the post-Webernian tradition elevated exactitude, abjuring not only Schoenberg's neo-Wagnerian rhetoric but also his interest in approximate pitch shapes. See, e.g., Boulez's (1968: 268) essay "Schoenberg Is Dead."

59 To some extent gestural listening is modeled by the theory of contour, but contour theory discards pitch entirely while assuming precise recognition of ordinal positions: B4–G♯4–G4–A4 is considered to have the same contour as E6–C2–B♭0–B4, namely, (3, 1, 0, 2), but a different contour from E4–C4–B♭3–B3, which is (3, 2, 0, 1). This last sequence is intuitively quite close to the first—indeed, Schoenberg's op. 11/1 presents one as a variant of the other—yet has a different contour. Unlike approximate set theory, contour theory discards questions about distance altogether; unlike a truly gestural approach, it asserts the analytical relevance of subtle differences of ordering. See Morris 1993.

indeed, the note E is not a third above or below any other note. By contrast, a different plunking of fingers might produce B–C♯–D♯–E–G, with the characteristically tertian 4+1 structure; this has the tertian voicing C♯–E–G–B–D♯ but no particularly quartal voicing. Some sets are close to multiple categories: 02458 is both a near cluster with a single augmented second, and an exactly tertian ninth chord (D–F–A♭–C–E).

Also interesting is the absence of equipollent chords. Since 7 is a prime number, every nonzero interval cycles through all seven notes in the equiheptatonic scale. Since every two-note equiheptatonic set is cyclic, every five-note set must be as well. Thus, there are only three equiheptatonic classes of pentachord, and they are all interval cycles; there is no room in the equiheptatonic universe for a separate equipollent category. Instead, each interval cycle is a gapped cycle in the other two categories: the cluster CDEFG (or 01234_7) is an incomplete stack of thirds DF•CEG, voiced (2, 2, 2, 2), and an incomplete stack of fourths E•DGCF, voiced (4, 3, 1, 3); the third-stack DFACE is an incomplete cluster CDEF•A, voiced (1, 1, 1, 1), and an incomplete fourth stack EAD•CF, voiced (2, 2, 4, 3); and the quartal EADGC is an incomplete cluster CDE•GA, voiced (1, 1, 1, 1), and an incomplete third stack D•ACEG, voiced (3, 1, 2, 1).[60] These possibilities are shown on the third staff of Figure 2.2. Cyclic structure is less a property of set-classes than of registral arrangement.

Here we encounter a fascinating phenomenon, the quasi-complementarity of approximate set-classes with sizes *n* and 7 – *n*: trichords and tetrachords, dyads and pentachords, the singleton and hexachords. There are equipollent trichords and tetrachords but not pentachords or hexachords. There are three approximate classes of dyads and pentachords, all cyclic, but only one class of unison and hexachord. From an exact twelve-tone perspective there would be no reason to expect any relationship of this sort; indeed, one would hardly expect this correspondence if one used only the chunking method. But if we understand the connection between approximate twelve-tone set theory and exact seven-tone set theory, we can see why it arises.

Once again, there are systematic transformations relating the close, tertian, and open voicings. Starting with close position we can displace the second lowest note up by an octave, or the second highest note down by an octave, to obtain the tertian voicing (2, 1, 1, 2); from there we can displace the central note up or down by an octave to produce an open-position voicing; to go in the opposite direction, we octave-displace an outer note so that it lies within the chord (Figure 6.1). The open-position voicings can also be internalized by conceiving approximate pentachords tonally; Figure 6.2 shows one way to think about the three approximate set-classes, each open-position voicing cycling through a series of chordal elements (e.g., fifth → root → fourth → seventh → third → [fifth]). This tonal perspective can be conceived as purely calculational, a tool for understanding

60 Julio Herrlein (2011: 162) observes that the pentachordal cluster can be voiced as a gapped stack of thirds.

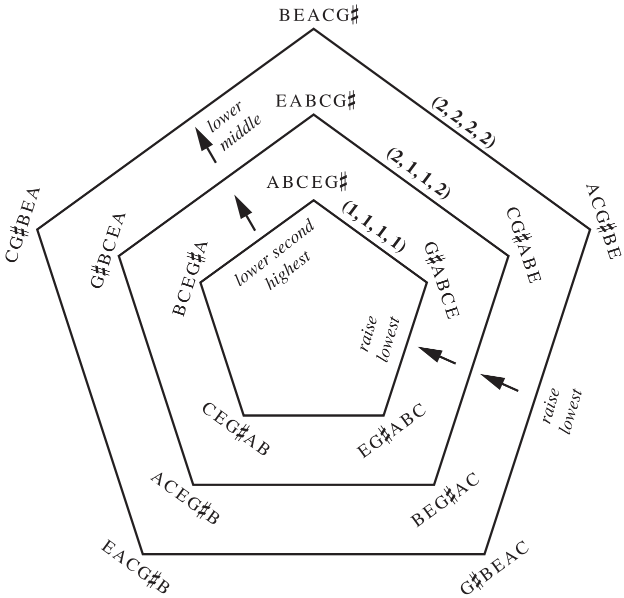

Figure 6.1A. A transformational graph relating pentachordal voicings. Moving along the pentagon transposes along the chord; moving between pentagons relates voicings by octave displacement.

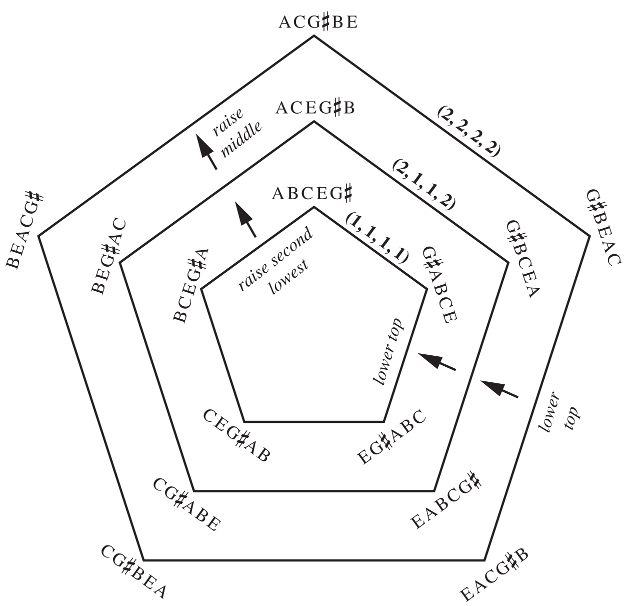

Figure 6.1B. A transformational graph relating pentachordal voicings, using different octave displacements.

Figure 6.2. One way to conceptualize the relations among open-position pentachordal voicings.

chromatic space; generally speaking, it is hard to comprehend chromatic space without such heuristics.

The pentachordal universe also contains a number of noncyclic sets, most of which have three- or four-note chromatic clusters that cannot be replicated within the seven-tone universe (see section 2); many are reasonably close to some cyclic category, as indicated by their subscripts. Others can be represented as gapped stacks of one sort but not the other; 01256, for example, is a near cluster with a single augmented second, and a singly gapped fifth stack **D**–[*g*]–**C♯**–**F♯**–**C**–**F**, but it is not a singly gapped third stack. Again, the surprise is how rare these noncyclic chords are: just five of the thirty-eight pentachords, if we assign near cycles to their closest category.

Concrete scale membership delivers similar results:

1. Clusters (8, scalar 01234_7): 01346, 01347, 01356, 01357, 01457, 02357, **02458**, 02468;
2. Tertian (7, scalar 01235_7): 01348, 01358, **01369**, 01458, 02358, **02458**, 02469;
3. Quartal (9, scalar 01245_7): 01368, **01369**, 01468, 01469, 01478, 01568, 02368, 02479;
4. Octatonic (1, scalar 01256_8): 01367;
5. None (16): 01234, 01235, 01236, 01237, 01245, 01246, 01247, 01248, 01256, 01257, 01258, 01267, 01268, 02346, 02347, 03458.

As in our previous categorization, 02458 is both clustered and tertian, and 01369 is both tertian and quartal. The None category contains all and only those pentachords with an 012 subset. The nearly quartal 01367 pentachord is anomalous, as it can only be embedded into the octatonic scale.

Categorizing pentachords by chromatic chunking gives five categories:

1. Clusters (10): 01234, 01235, 01245, 01246, 01346, 01356, 01357, 02346, 02357, 02468^{2};
2. Tertian (7): 01348, 01458, 01358^{1}, 02358^{1}, 02458^{1}, 01369^{3}, 02469^{13};
3. Quartal (5): 01267, 01268, 01568, 01368^{1}, 02479^{1};
4. Equipollent (2): 01457^{1}, 02368^{123};

	chunking	concrete scale	7tet
1. clusters	01234, 01235, 01245, 02346, 01346, 01356, 01357, 02357, 02468^{2}, 01246,	01346, 01356, 01357, 02357, 02468, 01457, 01347, **02458**	01346, 01356, 01357^{2}, 02357, **02468**^{3}, 01246, 01457, 01347^{2}, **02458**
2. tertian	01348, 01458, 01358^{1}, 02358^{1}, 02458^{1}, 01369^{3}, 02469^{13}	01348, 01458, 01358, 02358, **02458**, **01369**, 02469	01348, 01458, 01358, 02358^{1}, **02458**, **01369,** 02469^{3}, 01248, **02468**^{3}
3. quartal	01267, 01268, 01568, 01368^{1}, 02479^{1}	01568, 01368, 02479, 01468, 01469, 01478, 02368, **01369**	01268, 01568, 01368^{2}, 02479, 01468, 01469^{2}, 01478, 02368^{12}, **01369**
4. equipollent	01389^{1}, 02368^{123}		
noncyclic	01468^{13}, 01478^{3}, 01469^{23}, 01347^{1}, 01236^{1}, 01237, 01247^{1}, 01256^{1}, 01257^{1}, 02347^{1}, 03458^{2}, 01258^{2}, 01248^{2}, 01367^{13}	01236, 01237, 01247, 01256, 01257, 02347, 03458, 01258 01248, 01234, 01235, 01245, 01267, 02346, 01268, 01246	01236, 01237, 01247^{2}, 01256^{1}, 01257^{1}, 02347^{1}, 03458^{1}, 01258^{2}, 01367^{13}, 01234, 01235, 01245, 01267^{3} 02346^{1}

Figure 6.3. Comparing the different categorization systems: chunking, concrete scale, and equiheptatonic (7tet).

5. Noncyclic (14): 01237, 01236^{1}, 01247^{1}, 01256^{1}, 01257^{1}, 01347^{1}, 02347^{1}, 01367^{13}, 01468^{13}, 01478^{3}, 01248^{2}, 01258^{2}, 03458^{2}, 01469^{23}.

Once again, there is general agreement with the other methods (Figure 6.3). All but one of the noncyclic sets could be considered cyclic under a slight loosening of intervallic criteria. While the equipollent category has disappeared in the other two systems, it still exists here: the two equipollent pentachords are subsets of six-note clusters, tertian chords, and quartal chords.

7. Hexachords

With hexachords the story takes a surprising turn as the categories of cluster, tertian, and quartal merge. The underlying logic is again best illustrated by equiheptatonic quantization. A seven-note scale has only one six-note set-class, which is equally clustered, tertian, and quartal; this means that a set like CDEFGA can be arranged as a stack of seconds, thirds (**D** *e* **F** *g* **A C** *d* **E** *f* **G**), or fourths (**E** *f g* **A** *c* **D** *e f* **G** *a* **C** *d e* **F**). Our three categories are now one and the same.

The twelve hexachords closest to the six-note equiheptatonic set-class are all familiar: 013468, 013469, 013478, 013568, 013569, 013578, 013579, 014579, 023568, 023579, 023679, and 024579. These are *truly equipollent* in being equally clusters, stacks of thirds, and stacks of fourths. Smaller equipollent chords are gapped when arranged as stacks of seconds, thirds, and fourths; these hexachords can be shaped into ungapped cycles of each kind.[61]

In an ear-training context this is a significant shift. With smaller sets, the terms *cluster*, *tertian*, and *quartal* describe abstract pitch-class structure: a quartal trichord is one that can be arranged as a stack of fourth and tritones but can still sound quartal regardless of register; its saturation with fourths and tritones gives it a distinctive aural character. With hexachords these terms no longer divide set-class space into separate regions; instead, they describe ways of deploying one and the same group of hexachords. Thus, in smaller cardinalities terms like *quartal* have a double significance, referring both to internal intervallic constitution and to ways of arranging the chord in pitch. With hexachords the intrinsic meaning evaporates, leaving only the pitch-space meaning.

This prompts a more general speculation. If we believe that the approximate perspective captures something important about music perception, and if we think that the registral arrangement of six-note chords—attacked at once, as chords—is sometimes hard to perceive, then we may start to wonder whether hexachords are more effectively deployed linearly, since a melodic or arpeggiated configuration is more comprehensible than a simultaneous verticality. Here, in other words, we may start to see the transition from sets, or objects that can be easily conceived as unified gestalts, to scales, or reservoirs of pitch classes not typically present at any one musical moment. These two kinds of object are not always distinguished, in part because scales can be modeled formally as very large sets. But they are arguably quite different in their phenomenology.

Hexachords have three common cyclic voicings, shown on the bottom staff of Figure 2.2: the stack of intrinsic steps (1, 1, 1, 1, 1) (clustered or close position), the distorted stack of two-step intrinsic intervals (2, 2, 1, 2, 2) (the tertian voicing), and the distorted stack of three-step intrinsic intervals (3, 2, 3, 2, 3) (quartal).[62] The clustered and quartal voicings can be arranged in chains: a sequence of hexachordal steps contains all the close-position

61 Recall that the eleven-note chromatic chord is simultaneously a stack of semitones and perfect fourths; in much the same way, the six-note equiheptatonic set is simultaneously a stack of seconds, thirds, and fourths. Hexachords thus fail to satisfy what Quinn (2006) calls the "unique-genus property" since the equiheptatonic hexachord is simultaneously a paradigmatic cluster, tertian, and quartal chord: what are different genera at lower cardinalities merge in the case of hexachords.

62 Another cyclic voicing, (2, 1, 1, 1, 2) can be used to create stacks of seconds and thirds (see Figure A2 in the appendix). The voicing (2, 2, 3, 2, 2) is also reasonably common, offsetting the hexachord's two intrinsic 024_6 sets by three steps instead of one.

Figure 7.1. Clustered, tertian, and quartal voicings of the hexachord F♯–G–A–B♭–C–D. The clustered and quartal voicings can be arranged in overlapping chains. The letters C, T, and Q mark the most clustered, tertian, and quartal possibilities.

voicings, and a sequence of alternating three- and two-step hexachordal intervals contains all the quartal voicings (Figure 7.1). For a clustered hexachord, one of the (1, 1, 1, 1, 1) voicings is a stack of one- and two-semitone chromatic intervals; for a tertian hexachord, one of the (2, 2, 1, 2, 2) voicings is a stack of three- and four-semitone intervals; and for a quartal hexachord, one of the (3, 2, 3, 2, 3) voicings is a stack of five- and six-semitone intervals. Our discovery is that these properties tend to go together: typically, a clustered hexachord is also tertian and quartal. Figure 7.2 shows how the clustered, tertian, and quartal configurations appear in the opening measures of Berg's Violin Concerto; these voicings recur throughout the piece, which thematizes approximate interval cycles.

Once again, these facts are not fundamentally dependent on the equiheptatonic or any other scale. Suppose we have some hexachord, in any chromatic universe or even continuous unquantized space, that looks approximately like an equiheptatonic hexachord—which is to say, it has six approximately equal "small" steps and one large step that is approximately twice as big. Such chords can be arranged as clusters, stacks of thirds (i.e., stacks of intervals approximately the size of two small steps), or stacks of fourths (stacks of intervals approximately the size of three small steps). This is because we can arrange the tertian voicing (2, 2, 1, 2, 2) so that the one-step intrinsic interval is the large step, or the (3, 2, 3, 2, 3) voicing so that both two-step intrinsic intervals span the large step; in each case, the result is a fairly even voicing. Such possibilities are easier to see when we consider the equiheptatonic scale, but they are also available when we think chromatically.

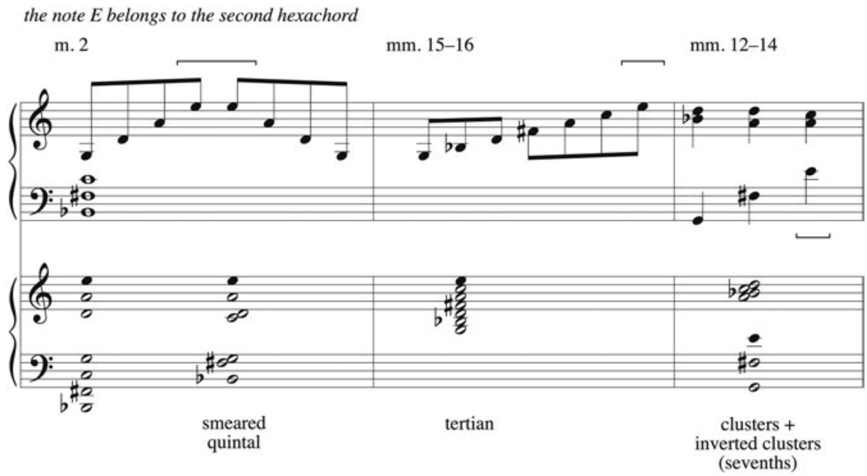

Figure 7.2. Quintal, tertian, and clustered voicings in Berg's Violin Concerto. In the last case, we have a chord progression whose efficient upper-voice counterpoint produces a cluster.

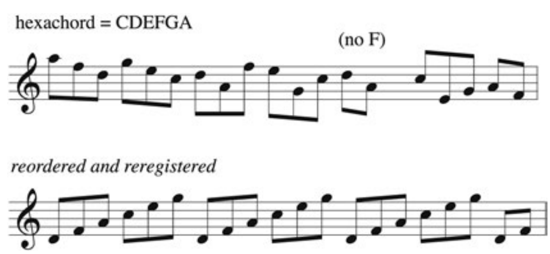

Figure 7.3. Improvising by alternating a hexachord's intrinsic 024_6 sets.

Our collection of twelve hexachords contains all of the tertian eleventh chords except the hexatonic collection. This fact underwrites an important improvisational idiom in which players alternate between the two component triads, reordering and occasionally omitting notes (Figure 7.3). This tactic represents a melodic expression of the hexachord's characteristically tertian voicing, (2, 2, 1, 2, 2), a relation that can be made manifest by reordering and reregistering each triad. Each triad is an 024_6 set contained within the hexachord considered as a six-note scale—that is, every other note of the hexachord, just as the whole-tone set contains every other note of the chromatic aggregate. The tertian voicing arranges the two 024_6 collections with one hexachordal step between them. For a tertian hexachord, this single intrinsic step can be chosen to be approximately the same chromatic size as the two-step intrinsic intervals within each triadic 024_6 subset. Hexachords are suspended between worlds, small enough to function as chords and large enough to be small scales.

The quartal voicing makes similar use of 03_6 hexachordal sets, each bisecting the hexachord the way the tritone bisects the chromatic scale. These are off-

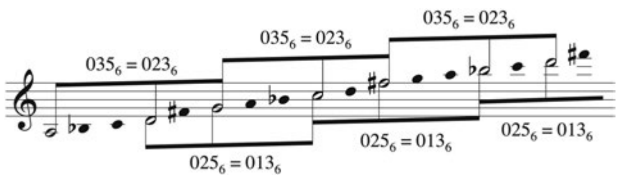

Figure 7.4. The quartal hexachordal voicing chains together hexachordal 023_6 and 013_6 sets.

Figure 7.5. Improvising using hexachordal 013_6 (top) and 012_6 (bottom) sets.

set to form the sequence (3, 2, 3, 2, 3), with each two-step interval containing the large hexachordal step. Figure 7.4 shows that this arrangement produces what David Lewin (1987) called a "retrograde inversional chaining" of the hexachord's 013_6 trichords; the chain passes through each of the hexachord's 013_6 and 023_6 sets and articulates all of its quartal voicings. This means that musicians can play 013_6 trichords (measured within the hexachord itself, considered as a scale) to produce mostly quartal resultants (measured chromatically; see Figure 7.5). Something similar can also be done with the hexachord's 012_6 clusters. Each of the hexachord's intrinsic trichordal set-classes can thus be associated with a cyclic voicing: 012_6 generates the clustered voicing, 024_6 the tertian voicing, and 013_6 the quartal voicing. The idioms in this paragraph partition the hexachord into nearly cyclic trichords, both when measuring within the hexachord and when measuring chromatically.

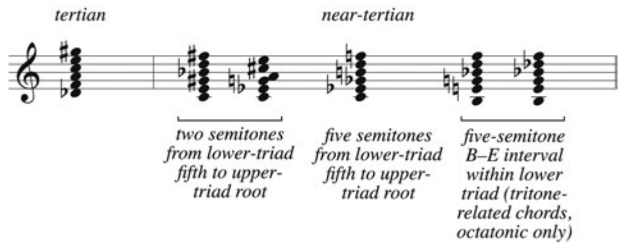

Figure 7.6. Tertian and near-tertian hexachords contained in the Pressing scales.

We can embed the fifty hexachords in the seven Pressing scales as follows:

1. Complete six-note scales (2): 02468A, 014589;
2. Six notes of a seven-note scale (13): 013468, 013469, 013478, 013479, 013568, 013569, 013578, 013579, 014579, 023568, 023579, 023679, 024579;
3. Purely octatonic subsets (3): 013467, 014679, 013679;
4. No embedding (32): 012345, 012346, 012347, 012348, 012356, 012357, 012358, 012367, 012368, 012369, 012378, 012456, 012457, 012458, 012467, 012468, 012469, 012478, 012479, 012567, 012568, 012569, 012578, 012579, 012678, 013457, 013458, 014568, 023457, 023458, 023468, 023469.

The first three categories can be described in two separate ways: they contain all the hexachords that do not contain consecutive semitones (an 012 chromatic cluster), and they are all the hexachords that can be formed by superimposing two triads (major, minor, diminished, or augmented).[63] Figure 7.6 shows that these augment the tertian hexachords of our first categorization with the remaining tertian hexachord (the hexatonic scale) and five near-tertian hexachords. This latter group includes two octatonic hexachords that are unique insofar as their two triadic roots are not hexachordally adjacent. All of these can be voiced in clustered, tertian, and quartal ways.

The chunking method provides the most nuanced view of the hexachordal universe:

1. Clusters (20): 012345, 012346, 012356, 012357, 012456, 012457, 013457, 013467, 023457, 023468^{2}, 02468A^{2}, 012467^{3}, 012468^{23}, **013468**^{3}, **013579**^{3}, **023568**^{3}, ***013568, 013578, 023579, 024579***;
2. Tertian (13): 014589, 013469^{13}, 013478^{13}, 013569^{13}, 014579^{13}, **013468**^{3}, **013579**^{3}, **023568**^{3}, **023679**^{1}, ***013568, 013578, 023579, 024579***;
3. Quartal (9): 012678, 012567^{1}, 012578^{1}, 012568^{12}, **023679**^{1}, ***013568, 013578, 023579, 024579***;
4. Equipollent (5): 012469^{123}, 013479^{123}, 014568^{123}, 013679^{13}, 014679^{13};

63 The Pressing scales contain all the nonchromatic hexachords that do not contain 012 subsets: one cannot form an 012 by superimposing triads; conversely, one can form all the nonchromatic hexachords by superimposing triads.

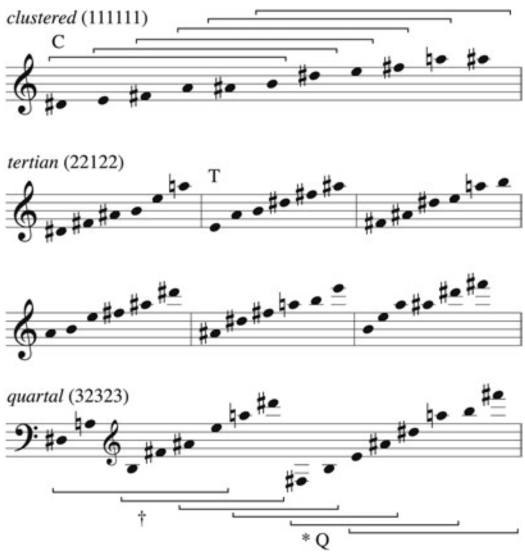

Figure 7.7. Clustered, tertian, and quartal voicings of the hexachord from Schoenberg's *Violin Concerto*. The asterisk and dagger mark the voicings used in Figure 1.8. The hexachord marked with the dagger appears inverted.

5. Noncyclic (15): 012348, 012369, 012378, 012347^{1}, 012358^{1}, 012367^{1}, 012368^{1}, 023458^{1}, 023469^{1}, 012458^{12}, 013458^{12}, 012478^{13}, 012479^{13}, 012579^{13}, 012569^{23}.

Boldface collections appear in two categories, while bold italic collections appear in three. As before, there is substantial overlap between categories, with four hexachords being clustered, tertian, and quartal, four hexachords belonging to two categories, and many others being close to multiple categories.

At first sight, this looks similar to the lists we have been considering, grouping hexachords into manageable and roughly equal-sized categories. But this impression is somewhat misleading, as the categories are more musically and psychologically fragile than their lower-cardinality analogues. First, it does not make sense to divide the hexachords into cluster, tertian, and quartal when so many hexachords belong to multiple groups. Second, because our ears are more tolerant as chords get larger, 012578 can sound reasonably clustered and tertian despite not being exactly so. With larger chords, such terms as *cluster*, *tertian*, and *quartal* are better understood as ways of deploying sets rather than as classifications of sets in themselves.

Figure 7.7 presents the basic cyclic voicings for the 012578 hexachord used in Schoenberg's violin concerto; in the preceding list it is categorized as quartal and nearly clustered. The figure identifies its maximally clustered, tertian, and quartal voicings: D♯–E–F♯–A–A♯–B (cluster), E–A–B–D♯–F♯–A♯ (tertian), and F♯–B–E–A♯–D♯–A (quartal, appearing in Figure 1.8 as a smeared variant). The quartal voicing is exact, while the clustered and tertian voicings are a semitone away from exactly qualifying. I am more interested in the fact that this hexachord can be made nearly clustered, tertian, and quartal than in the fact that the quartal voicing is just a tad more regular than the others. In other words, I consider the voicings in Figure 7.7 to be more or less equally good compositional starting points.

Many chord-classification schemes assume that complements share the same intrinsic quality, but this is not true of our method; for example, the hexachord 023568 is clustered, tertian, and quartal (or nearly so on the chunking method), while its complement, 023469, is noncyclic (though nearly clustered on the chunking method). Personally, I think this is a virtue, as I do not find complements to be aurally similar.[64] While there may be particular styles in which complements behave similarly, I do not think we should elevate this genre-specific fact to a general theoretical principle.[65] Nor should we let the desire for taxonomic simplicity override the possibility that very large chords might be more homogeneous than their complements; indeed, I find nine-note collections hard to distinguish when played as close-position simultaneities, whereas trichords are instantly recognizable. Complements can sound very different, and large collections require new compositional strategies.

8. Large sets

Approximate set theory breaks down as we consider larger and larger chords. Clearly, scalar embedding and equiheptatonic quantization will fail with large sets: a typical eight-note set will not belong to any familiar scale, and equiheptatonic quantization will send distinct chromatic pitch classes to the same equiheptatonic target.[66] Formally, however, the chunking method continues to work:

1. Clusters (26): 0123456, 0123457, 0123467, 0123567, 0123468, 0123568, 0124568, 0123578, 0124578, 0123579, 0234568, 0134568, 0234579, 0234679, 0134578, 0124678^{3}, 0124579^{2}, 0134579^{2}, 0135679^{23}, **0124679**^{2}, **0134679**^{3}, **012468A**^{3}, ***0124689, 0134689, 013468A, 013568A***;
2. Tertian (9): 0124589^{1}, **0134679**^{3}, **012468A**^{3}, **0125689**^{1}, **0124679**^{2}, ***0124689, 0134689, 013468A, 013568A***;
3. Quartal (9): 0123678^{1}, 0123679^{1}, 0125679^{1}, **0124679**^{2}, **0125689**^{1}, ***0124689, 0134689, 013468A, 013568A***;
4. Equipollent (3): 0123569^{1}, 0124569^{1}, 0145679^{1};
5. None (4): 0123458^{1}, 0123478^{1}, 0123469^{1}, 0123479^{1}.

Once again we see substantial overlap among the tertian and quartal categories: all but one tertian heptachord and all but three quartal heptachords belong to a second category. "Clusteredness" is becoming a default, with 68 percent (26 of 38)

64 Since a chord can be paradigmatically clustered while its complement fails to be, our three categorization schemes lack what Quinn (2006) calls the "prototype complementation property."

65 Forte (1972) argues that the presence of both a set and its complement indicates its structural significance; this means that the presence of complementation is not so much a discovery about a particular piece or style as it is an analytical ideal—a criterion of analytical goodness rather than an empirical observation.

66 In principle, this need not be considered problematic, but I find it counterintuitive.

of the heptachords being clustered compared with 40 percent (20 of 50) of the hexachords.

For octachords the domination by cluster becomes even more severe, with 86 percent (25 of 29) belonging to that category:

1. Clusters (25): 01234567, 01234568, 01234578, 01234579, 02345679, 01234678^{3}, 01234679^{3}, 01345679^{3}, 0124678A^{23}, **01235678**, **01235679**, **01245679**, **01235789**^{2}, **01245789**^{2}, **01234689**^{3}, **0123468A**^{3}, **0134679A**^{3}, ***01235689***, ***01245689***, ***0123568A***, ***0124568A***, ***0123578A***, ***0124578A***, ***01345689***, ***0134578A***;
2. Triads (12): 01234589^{1}, **01234689**^{3}, **0123468A**^{3}, **0134679A**^{3}, ***01235689***, ***01245689***, ***0123568A***, ***0124568A***, ***0123578A***, ***0124578A***, ***01345689***, ***0134578A***;
3. Quartal (14): 01236789^{1}, **01235678**, **01235679**, **01245679**, **01235789**^{2}, **01245789**^{2}, ***01235689***, ***01245689***, ***0123568A***, ***0124568A***, ***0123578A***, ***0124578A***, ***01345689***, ***0134578A***;
4. Equipollent: 01234789^{13};
5. None: 01234569^{1}.

I will not list the possibilities for the nonachords: all twelve can be expressed as stacks of seconds, and all but two can be expressed as stacks of thirds; half can be expressed as stacks of fourths.

I provide this information mainly to satisfy the reader's curiosity. My own belief is that for large chords the terms *clustered*, *tertian*, and *quartal* largely lose their utility as descriptors of intrinsic intervallic content; instead, they are better conceived as indications of how chords can be used. This may strike some readers as a weakness of approximate set theory, particularly when compared to alternatives that purport to offer universally valid systems of chord classification. But almost all of these alternatives require that listeners perceive large collections with a high degree of accuracy, keeping an exact count of every interval they hear.

The problem is that set theory's perceptual challenges are magnified when it comes to larger sets. Set theorists often tacitly assume that every set-class is equally distinctive and identifiable—an assumption that is problematic in the case of small sets and doubly so as sets grow larger.[67] My experience is instead that quality-space contracts as cardinality increases: while a few large collections might shine forth as particularly recognizable (e.g., the diatonic and octatonic scales), larger chords tend to be less distinct than smaller chords. The limitations of approximate set theory may therefore reflect something psychologically important; perhaps as collections grow they fade into the chromatic background.

In other words, composers need to work to make large collections sound special. This is in many ways a very familiar point. Numerous theorists have noticed that the principle of inversional equivalence breaks down as chords grow: in Gustav Mahler's or Debussy's work the notes C–E–G–A can function either as C major or A minor, and this means that chordal identity is no longer determined

67 This is most charitably described as an idealization. The question is how useful the idealization is (Tymoczko 2020a).

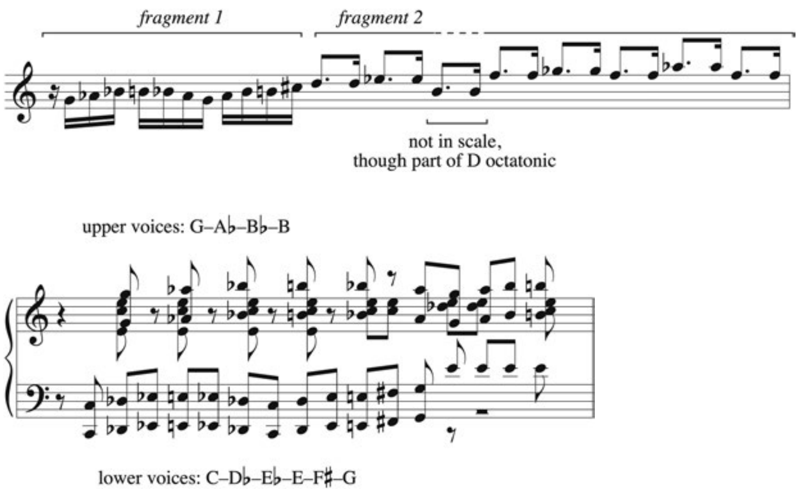

Figure 8.1. Shostakovich using time and register to articulate the different octatonic fragments of a nine-note scale in mm. 51–56 (A) and mm. 66–72 (B) of the Seventh String Quartet, III.

Figure 8.2. A Mahanthappa-style line partitioning of the 01347B hexachord into a major triad (open note heads) and a 024 subset. The line uses transpositions on C and F♯.

by pitch-class content; instead, voicing and pitch structure—and particularly the bass—play a crucial role. What I am suggesting is that something similar may occur in the atonal domain. There are many different ways to hear a large pitch-class set, with "undifferentiated," "gray," and "chromatic" being among the easiest.

Perhaps the most broadly useful strategy is to divide larger sets into (possibly overlapping) subsets that are separated registrally or temporally. Figure 8.1 shows Dmitri Shostakovich using this technique with a nine-note scale that is the minor triad's complement; he divides the collection into two octatonic subsets, each kept separate from the other. My colleague Rudresh Mahanthappa has developed a similar technique for working with hexachords: frequently, he breaks hexachords into a pair of trichords, moving back and forth between them but freely reordering each (Figure 8.2). This strategy is also characteristic of Barry Harris's playing, which decomposes nearly even eight-note scales into a stable tetrachord (usually a dominant seventh or added-sixth chord) and an embellishing diminished seventh, each containing every other note of the scale (Figure 8.3).

Figure 8.3. Barry Harris alternating between 0246 tetrachords of an eight-note scale, each decorated with an appoggiatura. The example comes from Bicket 2001.

These partially ordered collections are intermediate between the unordered collections of set theory and completely ordered twelve-tone rows. What is interesting is that they are both audible to the listener and flexible enough to be used in improvisation; in effect, large collections become little chord progressions.

Set theory, whether exact or approximate, is most useful when dealing with collections that can be heard as gestalts. This becomes harder and harder to do as collections grow: even six-note chords can be difficult to distinguish when presented as staccato chords. Larger collections thus tend to function as composites—scale-like sources of material, rarely present at any one moment. Exploring the boundary between these two musical regimes is a matter for another article.

9. Conclusion

Approximate set theory combines several interlocking ideas. Its most general claim is that music presents a broad spectrum of organizational possibilities. We are familiar with its exact end—the domain of canon and twelve-tone rows—where notes are rigidly interrelated. At the other extreme are pieces that ask us to adopt very general categories rooted in texture and gesture. Neither approximate nor exact set theory provides a unified description of all the chords of Figure 9.1, yet they are palpably similar, combining close-position sonorities in very high and low registers. Composers have made effective use of these sorts of categories, writing music in which the specific notes are less important than register, articulation, dynamics, and timbre. Music theory should not ignore this possibility just because it is difficult to formalize.

Somewhere toward the middle of the spectrum we find music that can be analyzed using the tools of approximate set theory. These tools include both the heuristic chord categories of cluster, tertian, quartal, and equipollent and a new form of exact set theory that measures intervals along the intrinsic scale. From this perspective every chord presents multiple sets simultaneously: a chromatic pitch-class set, a scalar pitch-class set, and a voicing measured along the intrinsic

Figure 9.1. A series of audibly similar sonorities, none of which are exactly related to the others.

scale. Attending to these different levels of structure reveals resemblances that might otherwise escape our notice. Rather than treating the "So What" and "Farben" chords as completely unrelated, we can say they are both open-position pentachords. Rather than saying "*Pierrot Lunaire* starts with a lot of thirds," we can say "Schoenberg uses the (2, 1, 2) voicing to bring out the equipollent tetrachord's tertian quality." This is an improvement not because it translates intuitive composerly practice into cryptic academic jargon (though that may have its advantages when it comes to tenure committees) but because it allows us to understand the particular techniques that constitute intuitive musical knowledge—helping us say exactly what composers are doing when they write a lot of thirds.

One attractive feature of this approach is that it creates affinities between sets of different sizes. The basic objects of Anton Webern's op. 7/3 are clusters or superimpositions of clusters (Figure 9.2). The accompanimental sets are all presented as transpositional combinations, one half of the chord transposing the other in pitch. We begin with a sequence of tetrachords whose semitonal pairs expand systematically: A–B♭–C♯–D, A–B♭–D–E♭, A♭–A–D–E♭.[68] We then hear a two-note cluster (the minor ninth), a three-note cluster (which I interpret, somewhat poetically, as a multiset that contracts the semitone pairs until they overlap), and a six-note approximate cluster. These collections are all voiced regularly as gapped and smeared stacks of seconds, thirds, and fourths: most of the voicings superimpose semitones, major sevenths, or minor ninths; the third and fourth chords, however, sublimate the clustered structure, combining semitone with major seventh. Meanwhile, the lyrical piano line presents a series of chromatic sets that are not regularly voiced or arranged as transpositional combinations; their freer registration perhaps reflects their status as primary melody (Figure 9.2B).[69] The melodic note B does not belong to any chromatic cluster,

68 This transpositionally symmetrical arrangement might be described as a Schoenbergian "motive of the accompaniment" (Schoenberg 1967: 83). As Richard Cohn (1988, 1991) emphasizes, these sets tend to be inversionally symmetrical. The initial expansion can be understood with the theory of K-nets; in my notation it involves the transpositions $\mathbf{T}_{1/2}$ and $\mathbf{T}_{-1/2}$ operating on equivalence classes of strongly isographic chords (Tymoczko 2007).

69 Lewin (1987, chap. 5) provides an exact set-theoretical analysis that is instructively different from mine.

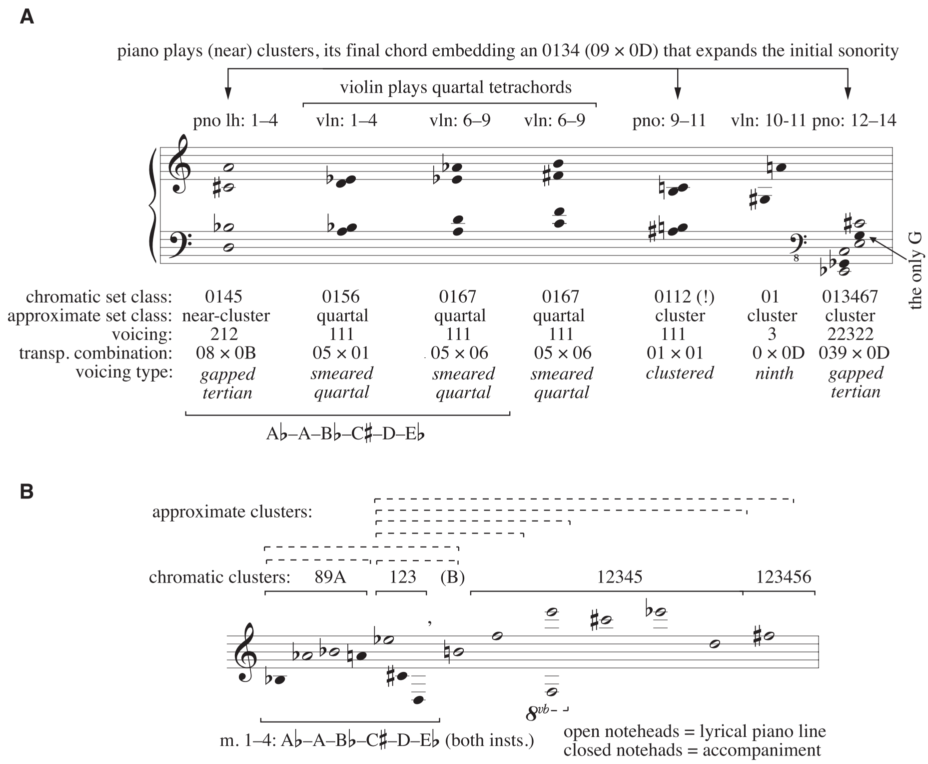

Figure 9.2. Webern's op. 7/3. A. The accompanimental collections used in the piece. B. The lyrical piano melody. In the "transp. combination" row the letters A, B, C, etc., refer to the numbers 10, 11, 12, etc.

but it does form an approximate cluster with all of the semitonal clusters in the melody.[70] Overall, the piece seems less concerned with set-theoretic exactitude than with more elusive similarities and differences: the affinity between clusters of various sizes, both chromatic and approximate; the contrast between the accompaniment's symmetrical voicings and the melody's asymmetrical registration; the resemblance between the piano's first sonority (a gapped tertian structure comprising two augmented triads a major seventh apart, each missing the same note) and its last (a gapped tertian structure comprising two diminished seventh chords a minor ninth apart, each missing the same note); and so on.

Approximate set theory thus stands at the intersection of many different kinds of thinking: tonal and nontonal, scalar and nonscalar, compositional and improvisational, pitch and pitch class. Many of the phenomena we have examined are easy to recognize when we are thinking inside a seven-note scale; in diatonic space, for example, it is clear that the pentachord 01245_7 is quartal (25140_7)

70 The lyrical melody combines with the violin accompaniment (chords 3 and 4 on Figure 9.2A) to present every chromatic note except G.

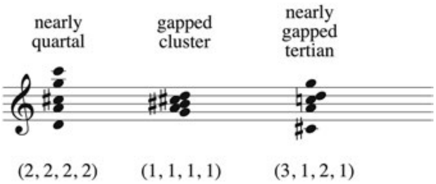

Figure 9.3. A nearly quartal pentachord, voiced as a stack of near fourths, as a gapped cluster, and as a gapped stack of near thirds.

while also being a gapped stack of seconds ($012{\bullet}45_7$) and thirds ($1{\bullet}5024_7$).[71] It is not obvious that similar possibilities would be available in the purely chromatic world. But consider the pentachord G–A–C–C♯–D, classified as noncyclic by each of the three systems discussed here. Figure 9.3 voices the chord as a stack of near fourths, D–A–C♯–G–C (which would be quartal were A lowered by semitone); as a gapped cluster, GA•CC♯D; and a gapped stack of near thirds, C♯•ACDG, which would be gapped tertian were D raised by semitone. Approximate set theory highlights these possibilities, repurposing seemingly tonal concepts as tools for understanding the chromatic domain.

This perspective also alerts us to an important change that occurs as collections grow: for smaller cardinalities, adjectives such as *tertian* and *quartal* can be taken either as descriptions of pitch-class content or as ways of arranging notes in register; with larger sets the latter meaning is primary. It makes relatively little sense to talk about tertian hexachords as opposed to quartal hexachords, for the very collections that can be arranged as stacks of thirds can typically also be arranged as stacks of fourths. By emphasizing the approximate nature of music perception, we draw attention to the point at which approximate categories break down. This is close to the boundary between chord and scale, the point at which we stop being able to conceive of musical objects as unified gestalts. Understanding this boundary, and the techniques composers use to work with larger collections, is a topic for future work.

About fifteen years ago, one of my teachers—an eminent composer educated in the European avant-garde tradition—heard me lecture about voice-leading geometry. After my talk he complained that I had talked only about pitch class and not about pitch. Being proud of my ideas and a little stung by his criticism, I responded by accusing him of being reflexively antitheoretical. "Don't you feel that the different voicings of a C-major triad sound noticeably similar," I asked, "at least so long as we avoid extremes of register and spacing?" (One should imagine this sentence illustrated by an irritating plunking out of C-major chords, each voiced differently.) Having thought about the matter for another decade and a

71 A number of diatonic theorists (e.g., Herrlein 2011) have noticed various phenomena discussed in this article.

half, I have come to appreciate his position: as chords get larger, arrangement in pitch starts to matter more than intrinsic pitch-class content. In other words, both of us had good points. With small cardinalities abstract pitch-class structure is often perceptible and important, but as chords grow, the relative priority of pitch and pitch class reverses. Exactly where this happens is a complex matter that can differ from listener to listener and from context to context. This leaves us in the challenging situation where there is no generally applicable model of pitch. Music is difficult because it requires us to balance overlapping and sometimes incompatible logics.

A basic challenge of twenty-first-century theory is the divergence between theory and practice. It is easy to frame this as a conflict between theorists and everyone else, and it is certainly true that theorists have made problematic assumptions about what is audible or aesthetically significant. But the divergence can also be seen within the activity of music making, in the gap between listeners' experience and the rigid structures manipulated by music makers—whether pitch-class sets, superimposed interval cycles, twelve-tone rows, spectral analyses of audio signals, Fibonacci numbers, or prolongational graphs. Approximate set theory aspires to bridge this gap, providing categories that are resilient to a certain degree of perceiver error. In this way it gestures toward a music theory that is both approximate and informative. We know how to do theory in an idealized environment where listeners accurately perceive every musical detail. Can we learn to do it in a way that forthrightly acknowledges human limitation?

10. Appendix

This appendix provides technical details about voicing, the Fourier transform, and the notion of evenness.

(a) Voicing

As explained in section 1, we can model voicings as intrinsic intervals measured between the adjacent notes of a chord, proceeding upward from the bottom. Given an n-note voicing $(x_1, x_2, \ldots, x_{n-1})$, we can reverse its intervals to form the *retrograde* $(x_{n-1}, x_{n-2}, \ldots, x_1)$, turning E3–C4–G4–A4–B4 or $(4, 2, 1, 1)$ into E3–G3–A3–C4–B4 or $(1, 1, 2, 4)$.[72] The *inversion* of a voicing $(-x_1, -x_2, \ldots, -x_{n-1})$ turns its intervals upside down so that the bottom note of the original chord becomes the top note of the inversion; this is the same as the retrograde if we order notes by pitch. If the intervals lie between unison and octave, then we can define a second notion of inversion that replaces each interval with its octave complement $(n - x_1, n - x_2, \ldots, n - x_{n-1})$; this turns E3–C4–G4–A4–B4 or

72 As Yust points out, reversing the intervals in an ordered series produces the "retrograde inversion" of twelve-tone theory; conversely, my use of *retrograde inversion* refers to the twelve-tone retrograde. The change in terminology arises because voicings are defined by intervals rather than as notes. This is also why an n-note voicing is represented by $n-1$ numbers.

Figure A1. The basic trichordal, tetrachordal, and pentachordal voicings, categorized by the standard twelve-tone operations. Cyclic voicings are shown with open note heads. Numbers show the spacing in intrinsic steps.

(4, 2, 1, 1) into E3–G3–C4–B4–A5 or (1, 3, 4, 4), with each interval *i* becoming 5 – *i*.[73] The *retrograde inversion* $(n - x_{n-1}, n - x_{n-2}, \ldots, n - x_1)$ is interesting because it can be used to retrograde a concrete voicing's pitch classes, turning E3–C4–G4–A4–B4 or (4, 2, 1, 1) into B2–A3–G4–C5–E5 or (4, 4, 3, 1). This turns clustered voicings into septimal voicings, tertian into sextal, and quartal into quintal. If we limit ourselves to voicings in which notes are less than one octave from their neighbors, and if we group voicings by the four twelve-tone operations, we obtain one class of trichordal voicing, three classes of tetrachordal voicing, eight classes of pentachordal voicing, and thirty-eight classes of hexachordal voicing (Figures A1, A2).[74] These are also the different tone-row classes of the associated low-cardinality universes.

A *cyclic voicing* is a voicing that can be used to express an exact interval cycle—that is, a voicing that can be embodied by a collection of pitches each the same distance above its lower neighbor. Cyclic voicings are retrograde symmetrical and shown on Figures A1 and A2 with open note heads.[75] If the voicing

73 Limiting intervals to less than an octave yields a variant of what Morris 1995 calls *PCINT equivalence*. There is nothing intrinsically wrong with voicings that have more than an octave between adjacent notes; when cataloguing voicings, however, it is helpful to consider them variants of their compact analogues.

74 Weber (1817–21) 1846: 184 (§63) identifies the voicings of three- and four-note chords. Harrison 2014 independently reconstructs Weber's list.

75 Since interval cycles are inversionally symmetrical in pitch, their spacing in intrinsic steps must be retrograde symmetrical.

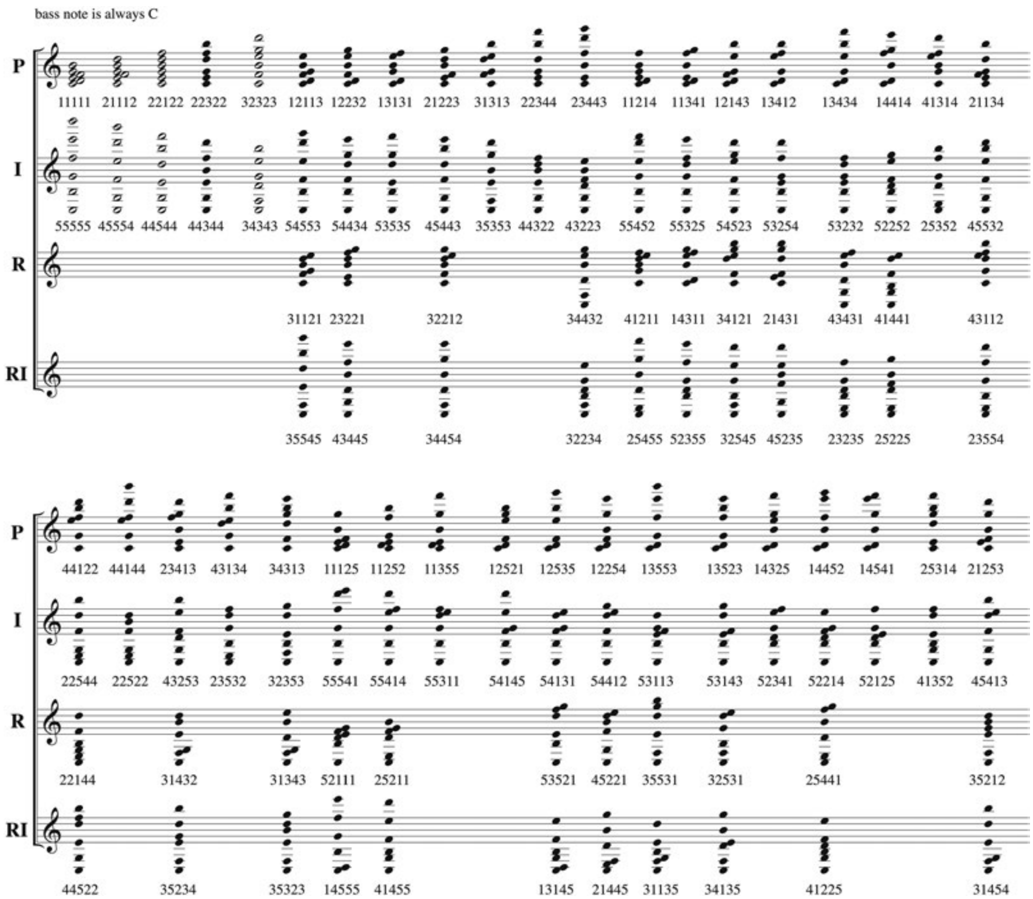

Figure A2. The basic hexachordal voicings.

$(x_1, x_2, \ldots, x_{n-1})$ is cyclic, then $(x_1 + n, x_2 + n, \ldots x_{n-1} + n)$ is also cyclic, as it simply adds an octave to the generating interval. If a cyclic voicing's generating interval is less than an octave, then its inversion $(n - x_1, n - x_2, \ldots, n - x_{n-1})$ is also cyclic. A *primitive* cyclic voicing is a cyclic voicing that can express an exact interval cycle whose generating interval lies between zero and half an octave. For an n-note chord, there are $n - 2$ primitive cyclic voicings of the form $(1, 1, \ldots, 1)$, $(2, 1, 1, \ldots, 1, 2)$, $(2, 2, 1, \ldots, 1, 2, 2)$, $(2, 2, \ldots, 2)$, $(3, 2, 2, \ldots, 2, 3)$, and so on. The boundaries between the cyclic voicings are determined by the values of g for which the interval between the chord's first and ith notes is an octave, with $2 \leq i \leq n$. Reading upward from the bottom of Figure 1.3, these are 6, 4, 3, 2.4, 2, . . . , or 12/2, 12/3, 12/4, 12/5, 12/6, ... A *well-formed* cyclic voicing has only one type of intrinsic interval; in Figure 1.3 these are close position $(1, 1, 1 \ldots)$, open position (2, 2 , . . .), and the unnamed (3, 3, . . .).[76] A *clustered* chord is a chord that can be voiced with each note one or two semitones above its lower neighbor; a *tertian* chord can be voiced with each note three or four semitones above its lower neighbor; and a *quartal* chord can be voiced with each note either five or six semitones above its lower neighbor. These categories roughly match the numbers on Figure 1.3, corresponding to $12/2 > g > 12/3$ (quartal), $12/3 > g > 12/4$ (tertian), and $12/6 \geq g$ (clustered). This is the mathematical link between intrinsic voicings and approximate set-class categories.

Throughout the article, I have made statements such as "for any pentachord, the open-position voicing will make it most nearly quartal." The theory of voice leading justifies these claims. Let us say that a chord is *completely quartal* when it is voiced as an exact stack of perfect fourths (or, generalizing the argument, some other interval). A completely quartal chord will use the relevant cyclic voicing on Figure 1.3. We can get a continuous measure of "fourthiness" (or, more generally, cyclicality) by calculating the voice-leading distance from any voicing to the nearest completely quartal voicing. It can be shown that for any reasonable measure of voice-leading size, there will always be a minimal voice leading that is "strongly crossing free"—connecting two chords spaced in the same pattern of intrinsic steps (Tymoczko 2011). Thus, we can minimize the distance from any pitch-class set to the nearest completely quartal chord by voicing it in the appropriate cyclic voicing.[77]

(b) Approximate interval cycles and the Fourier transform

Central to Ian Quinn's work is the proposal that the Fourier transform provides a continuous measure of chord quality (Quinn 2006, 2007).[78] Quinn observed that when chords are expressed as histograms (essentially bar graphs or vectors of weights assigned to pitch classes) the Fourier transform provides something like a continuous version of the interval vector: a six-element list of complex numbers representing chordal saturation with various intervals (its degree of "minor second-ish-ness," "major second-ish-ness," etc.).[79] This quantity can be interpreted geometrically as proximity to prototypes that are maximally saturated with the interval in question.

76 The term *well-formed* comes from Norman Carey and David Clampitt (1989), whose theory of well-formed scales is structurally similar to the theory of cyclic voicings.

77 This argument ignores the possibility of adding doublings, as in the quartal F–B♭–E–B♭–E♭ with doubled B♭. We can think of this as a pentachordal multiset in open position, **F** *b♭* **B♭** *e♭* **E** *f* **B♭** *b♭* **E♭**. For a fixed set of doublings, the argument in the text is valid.

78 The program has since been developed in Amiot 2016 and Yust 2016. Readers can visualize the Fourier transform using software written by Jennifer Harding: https://www.jenndharding.com/vectorcalculator.

79 Quinn (2006: 121) cautions against associating Fourier components with specific intervals (the "intervallic half-truth"), arguing that it is not possible in all tuning systems. However, it is possible if we allow generating intervals outside the scale: Quinn's primary counterexample, the fourth Fourier component in ten-tone equal-temperament, can be associated with a generating interval of 2.5 steps (the twelve-tone equal-tempered minor third).

There are many similarities between our approaches. Quinn and I both want to replace the isolated points of set theory with something more flexible: coarse-grained categories in my case and graded lists of qualia in his. Both of us conceive of these categories geometrically, as regions in a space—voice-leading space in my case, and the space of Fourier-component magnitudes in his. Both of us consider interval cycles to be prototypical, in my case because most equiheptatonic sets are cyclic and in his because of the mathematics of the Fourier transform; this means that both approaches recall the ideas of pre-set-theoretical writers such as Cowell, Hanson, and Persichetti. In both cases our categories extend across cardinalities to associate chords of different sizes. In both theories the numbers $12/i$ are important for positive integers i: in Quinn's because they represent the frequencies of his Fourier basis functions, in mine because they represent the boundaries between cyclic voicings on Figure 1.3. A main difference is that my theory involves a greater degree of approximation than Quinn's: his categories correspond to specific chromatic intervals such as "minor second"; mine correspond to generic intervals such as "second."[80]

One way to make Quinn's theory more approximate is to quantize equal-tempered sets to the equiheptatonic scale, taking the Fourier transform of the result; the discrete Fourier transform will then have three nontrivial components corresponding to the terms *clustered*, *quartal*, and *tertian*. Another possibility is to remain in twelve-tone equal temperament but ignore Fourier components beyond the third. Collections with a strong first component can be considered clustered, those with a strong second component quartal, and those with a strong third component tertian. Figure A3 shows that the resulting categorization is similar to the others in the text.[81] This illustrates a central Quinnian theme: the convergence of different approaches to chord categorization.

(c) Evenness and interval cycle

The property of evenness plays an important role in the theory of voice leading: the near equality of all the intrinsic intervals of one particular size (e.g., intrinsic thirds) allows transposition along a chord to nearly counteract transposition along the scale, thereby producing efficient voice leading (Tymoczko 2011, 2020b). Voicing gives us an alternate form of evenness that involves multiple types of intrinsic interval: in the quintic tetrachordal voicing C3–F♯3–C♯4–G4, the intervals C3–F♯3 and C♯4–G4 span two intrinsic steps, while F♯3–C♯4 spans three; what makes this chord quintic is that all of these are equal to six or seven

80 Since the Fourier transform is invertible, Quinn's interval vectors are exactly as fine-grained as those of traditional set theory. Quinn typically disregards phase, which has the effect of both ignoring transposition and grouping together "Z-related" or homometric sets. Nevertheless, the resulting theory is often closer to traditional set theory than to my approximate set theory; for example, on Quinn's view, complementary sets have similar qualia.

81 In making this list I intuitively chose a minimum threshold for a "strong" component; those failing to meet this threshold were considered equipollent.

	trichords	tetrachords
Clustered	012, 013, 024	0123, 0124, 0134, 0135, 0235, 0246
Quartal	016, 026, 027	0126, 0127, 0156, 0157, 0167, 0257, 0168
Tertian	037, 048	0145, 0148, 0158, 0248, 0347, 0358
Equipollent	014, 015, 025, 036	0125, 0136, 0137, 0146 0236, 0237, 0247

Figure A3. Categorizing small sets using the first three Fourier components.

semitones—and hence that it can be voiced as a nearly even fifth stack. Thus in the theory of *voice leading* we consider the near equality of intrinsic intervals all belonging to the same type (i.e., *n*-step intrinsic intervals); in the theory of *voicing*, we consider the near equality of intervals belonging to multiple types (i.e., *n*- and *m*-step intrinsic intervals, for two different numbers *m* and *n*).

We have also encountered another phenomenon familiar from voice-leading geometry: the reappearance of the same abstract structure at multiple hierarchical levels. The concrete voicing C3–E3–G♯3–B3–D♯4–G4 is a perturbed stack of major thirds whose intervals are (4, 4, 3, 4, 4) when measured semitonally—that is, an interval cycle in which the generating interval 4 is adjusted to avoid note repetition: $(x, x, x - 1, x, x)$.[82] It is also a cyclic voicing whose intrinsic spacing exhibits the same structure: the concrete voicing C3–E3–G♯3–B3–D♯4–G4 has intrinsic spacing (2, 2, 1, 2, 2), a sequence of two-step intrinsic intervals adjusted to avoid note repetition. Figure A4 shows that Quinn's "generic prototypes" can invariably be voiced in this way, as exact or minimally perturbed interval cycles both intrinsic and extrinsic.[83] Broadly speaking, he is interested in these chords' chromatic or extrinsic interval content, whereas I am interested in their intrinsic voicings; in other words, we generalize the same objects in different directions. Thus, Quinn places C–E–G♯–B–D♯–G and C–E♭–G♭–A–C♯–E in different categories, as prototypes

82 Quinn (2006) notes that many writers have been interested in perturbed interval cycles, including Hanson 1960, Eriksson 1986, and Headlam 1996.

83 Ordering by the generating interval (semitone, whole tone, major third, etc.) diverges from ordering by Fourier component (F1, F2, etc.). The characteristic voicing, shown in Figure A4, is determined by the former rather than the latter.

	CLUSTERED		TERTIAN		QUARTAL	
	Semitone (F1)	Wholetone (F6)	Minor third (F4)	Major third (F3)	Fourth (F5)	Tritone (F2)
2	C–C♯	C–D	C–E♭	C–E	C–F	C–F♯
3	C–C♯–D	C–D–E	C–E♭–F♯	C–E–G♯	C–F–B♭	C–F♯–B
4	C–C♯–D–D♯	C–D–E–F♯	C–E♭–F♯–A	C–E–G♯–B	C–F–B♭–E♭	C–F♯–B–F
5	C–C♯–D–D♯–E	C–D–E–F♯–G♯	C–E♭–F♯–A–C♯	C–E–G♯–B–D♯	C–F–B♭–E♭–A♭	C–F♯–B–F–B♭
6	C–C♯–D–D♯–E–F	C–D–E–F♯–G♯–A♯	C–E♭–F♯–A–C♯–E	C–E–G♯–B–D♯–G	C–F–B♭–E♭–A♭–D♭	C–F♯–B–F–B♭–E

Figure A4. Quinn's primary and secondary generic prototypes for dyads through hexachords. All are interval cycles or near-interval cycles. The numbers in the leftmost column represent chord-size. For unshaded cells the characteristic voicing is close position. For lightly shaded cells it is (2, 1, . . . , 1, 2). For dark-celled pentachords the characteristic voicing is open position (2, 2, . . . , 2); for dark-celled hexachords it is (3, 2, 3, 2, 3).

of the major- and minor-third genera, respectively; I place them in the same category, as tertian hexachords voiced (2, 2, 1, 2, 2). (This is reflected on Figure A4 by the larger categories, cluster, tertian, and quartal, each grouping two different columns.) These different interpretations are possible because Quinn's prototypes can be voiced so as to exhibit the same abstract structure on multiple levels—minimally perturbed interval cycles (x, x, $x\pm1$, x, x) whether we consider extrinsic semitones or intrinsic steps. As above, so below: this reappearance of similar structure at multiple hierarchical levels is one of the deepest and most mysterious features of Western music.

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